THE GROWTH RATE OF $\Gamma^3$ DEFINED BY ORLICZ FUNCTION

DEEPMALA$^1$, N. SUBRAMANIAN$^2$, AND LAKSHMI NARAYAN MISHRA$^3$

ABSTRACT. In this paper we introduce the growth rate of $\Gamma^3$ defined by modulus function and study general properties of these spaces and also establish some inclusion results and duals among them.

1. INTRODUCTION

Throughout $w$, $\Gamma$ and $\Lambda$ denote the classes of all, entire and analytic scalar valued single sequences, respectively.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [2], Subramanian et al. [3-9], and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [10], Esi et al. [11-15], Subramanian et al. [16-25] and many others.

Let $(x_{mnk})$ be a triple sequence of real or complex numbers. Then the series \( \sum_{m,n,k=1}^{\infty} x_{mnk} \) is called a triple series. The triple series \( \sum_{m,n,k=1}^{\infty} x_{mnk} \) give one space is said to be convergent if and only if the triple sequence \( (S_{mnk}) \) is convergent, where

\[
S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m, n, k = 1, 2, 3, \ldots) .
\]

A sequence $x = (x_{mn})$ is said to be triple analytic if

\[
\sup_{m,n,k} \frac{|x_{mnk}|^{\frac{1}{m+n+k}}}{m+n+k} < \infty.
\]

Key words and phrases. analytic sequence, modulus function, double sequences, \( \chi \) sequence, duals, rate spaces.

2010 Mathematics Subject Classification. 40A05; 40C05; 46A45; 03E72.

Received: 
Revised: 

dmrai23@gmail.com, deepmaladm23@gmail.com.
nsmaths@yahoo.com.
lakshminarayanimishra04@gmail.com.
The vector space of all triple analytic sequences are usually denoted by \( \Lambda^3 \). A sequence \( x = (x_{m,n,k}) \) is called triple entire sequence if
\[
|x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \quad \text{as} \quad m, n, k \to \infty.
\]
The vector space of all triple entire sequences are usually denoted by \( \Gamma^3 \). Let the set of sequences with this property be denoted by \( \Lambda^3 \) and \( \Gamma^3 \) is a metric space with the metric
\[
d(x, y) = \sup_{m,n,k} \{ |x_{m,n,k} - y_{m,n,k}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \ldots \},
\]
for all \( x = \{x_{m,n,k}\} \) and \( y = \{y_{m,n,k}\} \) in \( \Gamma^3 \). Let \( \phi = \{\text{finite sequences}\} \).

Consider a triple sequence \( x = (x_{m,n,k}) \). The \((m, n, k)\)th section \( x^{[m,n,k]} \) of the sequence is defined by
\[
x^{[m,n,k]} = \sum_{i,j,q} x_{ijq} \delta_{ijq} \quad \text{for all} \quad m, n, k \in \mathbb{N},
\]
where \( \delta_{mnk} \) is a three dimensional matrix with 1 in the \((m, n, k)\)th position and zero otherwise.

Let \( M \) and \( \Phi \) are mutually complementary Orlicz functions. Then, we have:
(i) For all \( u, y \geq 0 \),
\[
uy \leq M(u) + \Phi(y), \quad (\text{Young’s inequality}) \quad \text{[See[26]]}
\]
(ii) For all \( u \geq 0 \),
\[
\eta(u) = M(u) + \Phi(\eta(u)).
\]
(iii) For all \( u \geq 0 \), and \( 0 < \lambda < 1 \),
\[
M(\lambda u) \leq \lambda M(u)
\]
Lindenstrauss and Tzafriri [27] used the idea of Orlicz function to construct Orlicz sequence space
\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},
\]
The space \( \ell_M \) with the norm
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \((1 \leq p < \infty)\), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{m,n,k}) \) of Orlicz function is called a Musielak-Orlicz function. A sequence \( g = (g_{m,n,k}) \) defined by
is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak Orlicz function \( f \), the Musielak-Orlicz sequence space \( t_f \) is defined as follows

\[
t_f = \left\{ x \in w^3 : M_f (|x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty \right\},
\]

where \( M_f \) is a convex modular defined by

\[
M_f (x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} (|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric

\[
d(x, y) = \sup_{mnk} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left( \frac{|x_{mnk}|^{1/m+n+k}}{m_{nk}} \right) \right) \leq 1 \right\}.
\]

We consider \( t_f \) equipped with the Luxemburg metric space, (i.e)

Let \( (X_i, d_i), i \in I \) be a family of metric spaces such that each two elements of the family are disjoint. Denote \( X = \bigcup_{i \in I} X_i \). If we define

\[
d(x, y) = \begin{cases} d_i (x, y), & \text{if } x, y \in X_i \\ +\infty & \text{if } x \in X_i, y \in X_j, i \neq j \end{cases}
\]

then the pair \( (X, d) \) is a Luxemburg metric space.

A modulus function \( M \) is said to satisfy the \( \Delta_2 \)– condition for small \( u \) or at 0 if

\[
\text{for each } k \in \mathbb{N}, \text{there exist } R_k > 0 \text{ and } u_k > 0 \text{ such that } M (ku) \leq R_k M (u) \text{ for all } u \in (0, u_k].
\]

Moreover, an modulus function \( M \) is said to satisfy the \( \Delta_2 \)– condition if and only if

\[
\lim_{u \to 0^+} \sup M(2u) < \infty.
\]

If \( X \) is a sequence space, we give the following definitions:

(i) \( X' = \) the continuous dual of \( X \);

(ii) \( X^\alpha = \{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk}x_{mnk}| < \infty, \text{for each } x \in X \} \);

(iii) \( X^\beta = \{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk}x_{mnk} \text{ is convergent, for each } x \in X \} \);

(iv) \( X^\gamma = \{ a = (a_{mnk}) : \sup_{mnk \geq 1} \left| \sum_{m,n,k=1}^{M,N,K} a_{mnk}x_{mnk} \right| < \infty, \text{for each } x \in X \} \);

(v) \( \text{let } X \text{ be an } FK - \text{space } \supset \phi; \text{ then } X^f = \{ f(S_{mnk}) : f \in X' \} \);

(vi) \( X^\delta = \{ a = (a_{mnk}) : \sup_{mnk} |a_{mnk}x_{mnk}|^{1/m+n+k} < \infty, \text{for each } x \in X \} \);
\(X^\alpha, X^\beta, X^\gamma\) are called \(\alpha - (or \text{K"othe - Toeplitz})\) dual of \(X, \beta - (or \text{generalized - K"othe - Toeplitz})\) dual of \(X, \gamma - \text{dual of} \ X, \delta - \text{dual of} \ X\) respectively.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [28] as follows

\[Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}\]

for \(Z = c, c_0\) and \(\ell_\infty\), where \(\Delta x_k = x_k - x_{k+1}\) for all \(k \in \mathbb{N}\).

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

\[Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}\]

where \(Z = \Lambda^2, \chi^2\) and \(\Delta x_{mn} = (x_{mn} - x_{m+1n} - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}\) for all \(m, n \in \mathbb{N}\).

Let \(w^3, \chi^3(\Delta_{mnk}), \Lambda^3(\Delta_{mnk})\) be denote the spaces of all, triple gai difference sequence space and triple analytic difference sequence space respectively and is defined as

\[\Delta^m x_{mn} = \Delta \Delta^{m-1} x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{m+1n+1} - \Delta^{m-1} x_{m+1n+2} - \Delta^{m-1} x_{m+2n} - \Delta^{m-1} x_{m+2n+1} - \Delta^{m-1} x_{m+2n+2}\]

2. Definition and Preliminaries

A modulus function was introduced by Nakano [29]. We recall that a modulus \(f\) is a function from \([0, \infty) \to [0, \infty)\), such that

1. \(f(x) = 0\) if and only if \(x = 0\)
2. \(f(x + y) \leq f(x) + f(y)\), for all \(x \geq 0, y \geq 0\),
3. \(f\) is increasing,
4. \(f\) is continuous from the right at 0. Since \(|f(x) - f(y)| \leq f(|x - y|)\), it follows from here that \(f\) is continuous on \([0, \infty)\).

2.1. Definition. A sequence \(t\) is called a triple growth entire sequence of Orlicz, for a set \(X\) of sequences if \((x_{mnk}) = o(t_{mnk}) \Leftrightarrow f \left( \left| \frac{x_{mnk}}{t_{mnk}} \right|^{1/m+n+k} \right) \to 0\) as \(m, n, k \to \infty\).

2.2. Definition. A sequence \(t\) is called a triple growth analytic sequence of Orlicz, for a set \(X\) of sequences if \((x_{mnk}) = O(t_{mnk}) \Leftrightarrow f \left( \left| \frac{x_{mnk}}{t_{mnk}} \right|^{1/m+n+k} \right) < \infty \ \forall m, n, k\).
3. Main Results

3.1. Theorem. If $\Gamma^3_3$ has a growth sequence of Orlicz then $\Gamma^3_{f_3}$ has a growth sequence of Orlicz.

Proof: Let $\Gamma^3_{f_3}$ be a growth sequence of Orlicz. Then $f \left( \left| \frac{x_{mnk}}{\pi_{mnk}} \right| \right)^{1/m+n+k} \rightarrow 0$ as $m, n, k \rightarrow \infty$. Let $x \in \Gamma^3_{f_3}$. Then $\left\{ \frac{x_{mnk}}{\pi_{mnk}} \right\} \in \Gamma^3_f$. We have $f \left( \left| \frac{x_{mnk}}{\pi_{mnk}t_{mnk}} \right| \right)^{1/m+n+k} \leq f \left( |x_{mnk}| \right)^{1/m+n+k} \leq |\pi_{mnk}t_{mnk}| \rightarrow 0$ as $m, n, k \rightarrow \infty$, which means that $f \left( |x_{mnk}| \right)^{1/m+n+k} \leq |\pi_{mnk}t_{mnk}| \rightarrow 0$ as $m, n, k \rightarrow \infty$. Thus $\{\pi_{mnk}t_{mnk}\}$ is a growth sequence for $\Gamma^3_{f_3}$. In other words, $\Gamma^3_{f_3}$ has the growth sequence $\pi t$.

3.2. Theorem. Let $\Gamma^3_f$ be a BK metric space. Then the rate space $\Gamma^3_{f_3}$ has a growth sequence of Orlicz.

Proof: Let $x \in \Gamma^3_{f_3}$. Then $\left\{ \frac{x_{mnk}}{\pi_{mnk}} \right\} \in \Gamma^3_f$. Put $P_{mnk}(x) = \frac{x_{mnk}}{\pi_{mnk}} \forall x \in \Gamma^3_{f_3}$. Then $P_{mnk}$ is a continuous functional on $\Gamma^3_{f_3}$. Hence $|P_{mnk}| \rightarrow 0$ as $m, n, k \rightarrow \infty$.

Also for every positive integer $m, n, k$ we have $f \left( \left| \frac{x_{mnk}}{\pi_{mnk}} \right| \right)^{1/m+n+k} = |P_{mnk}(x)| \leq |P_{mnk}(x)| \pi_{mnk} f \left( |x_{mnk}| \right)^{1/m+n+k} \rightarrow 0$ as $m, n, k \rightarrow \infty$. Hence $x_{mnk} = o(P_{mnk} \pi_{mnk})$.

Thus $\{P_{mnk} \pi_{mnk}\}$ is a growth sequence for $\Gamma^3_{f_3}$.

3.3. Theorem. $(\Gamma^3_3)^{1/3} = \Lambda^3_{1/3}$

Proof: Let $x \in \Lambda^3_{1/3}$. Then there exists $M > 0$ with $|\pi_{mnk}x_{mnk}| \leq M^{m+n+k} \forall m, n, k \geq 1$. Choose $\epsilon > 0$ such that $\epsilon M < 1$.

If $y \in \Gamma^3_3$, we have $\left( \frac{y_{mnk}}{\pi_{mnk}} \right) \leq \epsilon^{m+n+k} \forall m, n, k \geq m_0n_0k_0$ depending on $\epsilon$.

Therefore $\sum |x_{mnk}y_{mnk}| \leq \sum (M \epsilon)^{m+n+k} < \infty$, Hence

$$\Lambda^3_{1/3} \subset (\Gamma^3_3)^{\alpha}$$

On the other hand, let $x \in (\Gamma^3_3)^{1/3}$. Assume that $x \notin \Lambda^3_{1/3}$. Then there exists an increasing sequence $\{p_{mnk}q_{mnk}\}$ of positive integers such that $|\pi_{p_{mnk}q_{mnk}}x_{p_{mnk}q_{mnk}}| > (m + n + k)^3(p_{mnk}q_{mnk}) \forall m, n, k > m_0n_0k_0$. Take Take $y = \{y_{mnk}\}$ by

$$y_{mnk} = \begin{cases} \frac{\pi_{mnk}}{(m+n+k)^3(p_{mnk}+q_{mnk})}, & \text{for } (p, q, r) = (p_m, q_n, r_k) \\ 0, & \text{for } (p, q, r) \neq p_mq_nr_k \end{cases}$$

Then $\{y_{mnk}\} \in \Gamma^3_3$, but $\sum |x_{mnk}y_{mnk}| = \infty$, a contradiction. This contradiction shows
that
\[(3.3) \quad (\Gamma^2_\pi)^\alpha \subset \Lambda^3_{1/\pi}\]
From (3.1) and (3.3) it follows that \((\Gamma^3_\pi)^\alpha = \Lambda^3_{1/\pi}\).

3.4. Theorem. \([\Lambda^3_{f_\pi}]^\beta = [\Lambda^3_{f_\pi}]^\alpha = [\Lambda^3_{f_\pi}]^\gamma = \eta^{3}_{M_\pi},\)
where \(\eta^{3}_{M} = \bigcap_{N \in N-N(1)} \left\{ x = x_{mnk} : \sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{|y_{mnk}|^{1/m+n+k}} \right) \right) \right\} < \infty \).

Proof (1) First we show that \(\eta^{3}_{f_\pi} \subset [\Lambda^3_{f_\pi}]^\beta\).

Let \(x \in \eta^{3}_{f_\pi}\) and \(y \in \Lambda^3_{f_\pi}\). Then we can find a positive integer \(n_0\) such that \(|y_{mnk}|^{1/m+n+k} < \max \left(1, \sup_{m,n,k \geq 1} \left( |y_{mnk}|^{1/m+n+k} \right) \right) < n_0\), for all \(m, n, k\).

Hence we may write
\[
\left| \sum_{m,n,k} x_{mnk} y_{mnk} \right| \leq \sum_{m,n,k} |x_{mnk} y_{mnk}| \leq \sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{|y_{mnk}|^{1/m+n+k}} \right) \right) \leq \sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{\frac{|y_{mnk}|^{1/m+n+k}}{\rho}} \right) \right).
\]
Since \(x \in \eta^{3}_{f_\pi}\), the series on the right side of the above inequality is convergent, whence \(x \in [\Lambda^3_{f_\pi}]^\beta\). Hence \(\eta^{3}_{f_\pi} \subset [\Lambda^3_{f_\pi}]^\beta\).

Now we show that \([\Lambda^3_{f_\pi}]^\beta \subset \eta^{3}_{f_\pi}\).

For this, let \(x \in [\Lambda^3_{f_\pi}]^\beta\), and suppose that \(x \notin \Lambda^3_{f_\pi}\). Then there exists a positive integer \(n_0 > 1\) such that \(\sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{\frac{|y_{mnk}|^{1/m+n+k}}{\rho}} \right) \right) = \infty\).

If we define \(y_{mnk} = \left( \frac{N^{m+n+k}}{|\pi_{mnk}!nmk|} \right) Sgn (x_{mnk}) m, n, k = 1, 2, \cdots\), then \(y \in \Lambda^3_{f_\pi}\).

But, since
\[
\left| \sum_{m,n,k} x_{mnk} y_{mnk} \right| = \sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{|y_{mnk}|^{1/m+n+k}} \right) \right) = \sum_{m,n,k} \left( f \left( \frac{|x_{mnk}|}{\frac{|y_{mnk}|^{1/m+n+k}}{\rho}} \right) \right) = \infty,
\]
we get \(x \notin [\Lambda^3_{f_\pi}]^\beta\), which contradicts the assumption \(x \in [\Lambda^3_{f_\pi}]^\beta\). Therefore \(x \in \eta^{3}_{f_\pi}\). Therefore \([\Lambda^3_{f_\pi}]^\beta = \eta^{3}_{f_\pi}\).

(ii) and (iii) can be shown in a similar way of (i). Therefore we omit it.
3.5. **Theorem.** If $\Gamma_{f\pi}^3$ is a growth sequence then $\eta_{f\pi}^3 \subset [\Gamma_{f\pi}^3]^{\beta} \subset \Lambda_{f\pi}^3$.

**Proof** Case 1: First we show that $\eta_{f\pi}^3 \subset [\Gamma_{f\pi}^3]^{\beta}$.

We know that $\Gamma_{f\pi}^3 \subset \Lambda_{f\pi}^3$.

$[\Lambda_{f\pi}^3]^{\beta} \subset [\Gamma_{f\pi}^3]^{\beta}$. But $[\Lambda_{f\pi}^3]^{\beta} = \eta_{f\pi}^3$, by Theorem 3.4.

Therefore

(3.4) $\eta_{f\pi}^3 \subset \Gamma_{f\pi}^3$.

**Case 2:** Now we show that $[\Gamma_{f\pi}^3]^{\beta} \subset \Lambda_{f\pi}^3$.

Let $y = \{y_{mnk}\}$ be an arbitrary point in $(\Gamma_{f\pi}^3)^{\beta}$. If $y$ is not in $\Lambda_{f\pi}^3$, then for each natural number $q$, we can find an index $m_qn_qk_q$ such that

\[
\left( f \left( \frac{|y_{m_qn_qk_q}/\pi_{m_qn_qk_q}|^{1/m_q+n_q+k_q}}{p} \right) \right) > q, \quad (1, 2, 3, \cdots)
\]

Define $x = \{x_{mnk}\}$ by

\[
\left( f \left( \frac{|x_{mnk}/\pi_{mnk}|^{1/m+n+k}}{p} \right) \right) = \frac{1}{q_{m+n+k}} \quad \text{for} \quad (m, n, k) = (m_qn_qk_q) \quad \text{for some} \quad q \in \mathbb{N}; \quad \text{and} \quad \left( f \left( \frac{|x_{mnk}/\pi_{mnk}|^{1/m+n+k}}{p} \right) \right) = 0, \quad \text{otherwise}.
\]

Then $x$ is in $\Gamma_{f\pi}^3$, but for infinitely $mnk$,

(3.5) $\left( f \left( \frac{|y_{mnk}/x_{mnk}|}{\rho} \right) \right)^{p_{mnk}} > 1$.

Consider the sequence $z = \{z_{mnk}\}$, where

\[
\left( f \left( \frac{3|z_{111}/x_{111}|}{\rho} \right) \right) = \left( f \left( \frac{3|z_{111}/x_{111}|}{\rho} \right) \right) - s
\]

with $s = \sum \left( f \left( \frac{x_{mnk}}{\rho} \right) \right)$; and

\[
\left( f \left( \frac{z_{mnk}/x_{mnk}}{\rho} \right) \right) = \left( f \left( \frac{z_{mnk}/x_{mnk}}{\rho} \right) \right) (m, n, k = 1, 2, 3, \cdots)
\]

Then $z$ is a point of $\Gamma_{f\pi}^3$. Also \( \sum \left( f \left( \frac{z_{mnk}/x_{mnk}}{\rho} \right) \right) = 0 \). Hence $z$ is in $\Gamma_{f\pi}^3$.

But, by the equation (3.5), \( \sum \left( M \left( \frac{z_{mnk}/x_{mnk}}{\rho} \right) \right) \) does not converge.

$\Rightarrow \sum x_{mnk}y_{mnk}$ diverges.
Thus the sequence $y$ would not be in $(\Gamma_{\pi}^3)^{\beta}$. This contradiction proves that
\begin{equation}
(\Gamma_{\pi}^3)^{\beta} \subseteq \Lambda_{\pi}^3.
\end{equation}

Choose $f = \text{id}$, where $\text{id}$ is the identity function and $y_{1nk}/\pi_{1nk}t_{1nk} = x_{1nk}/\pi_{1nk}t_{1nk} = 1$ and $y_{mnk}/\pi_{mnk}t_{mnk} = x_{mnk}/\pi_{mnk}t_{mnk} = 0$ $(m > 1)$ for all $n, k$, then obviously $x \in \Gamma_{\pi}^3$ and $y \in \Lambda_{\pi}^3$, but $\sum_{m,n,k=1}^{\infty} x_{mnk}y_{mnk} = \infty$, hence
\begin{equation}
y \notin (\Gamma_{\pi}^3)^{\beta}.
\end{equation}
From (3.6) and (3.7) we are granted
\begin{equation}
(\Gamma_{\pi}^3)^{\beta} \subseteq \Lambda_{\pi}^3.
\end{equation}

Hence (3.4) and (3.8) we are granted $\eta_{\pi}^3 \subset [\Gamma_{\pi}^3]^{\beta} \subset \Lambda_{\pi}^3$.

3.6. Proposition. $\Gamma_{\pi}^3 \subset \Gamma_{\pi}^3$

Proof: Let $x \in \Gamma_{\pi}^3$.

Then we have $\left| \frac{x_{mnk}}{\pi_{mnk}t_{mnk}} \right|^{1/m+n+k} \to 0 as m, n, k \to \infty$.

Here, we get $\left| \frac{x_{mnk}}{\pi_{mnk}t_{mnk}} \right|^{1/m+n+k} \to 0 as m, n, k \to \infty$. Thus we have $x \in \Gamma_{\pi}^3$ and so $\Gamma_{\pi}^3 \subset \Gamma_{\pi}^3$.

3.7. Theorem. If $\Gamma_{\pi}^3$ is a growth sequence then $\eta_{\pi}^3 \subset [\Gamma_{\pi}^3]^{\beta} \subset \Lambda_{\pi}^3$.

Proof Case 1: First we show that $\eta_{\pi}^3 \subset [\Gamma_{\pi}^3]^{\beta}$.

We know that $\Gamma_{\pi}^3 \subset \Lambda_{\pi}^3$.

$[\Lambda_{\pi}^3]^{\beta} \subset [\Gamma_{\pi}^3]^{\beta}$. But $[\Lambda_{\pi}^3]^{\beta} = \eta_{\pi}^3$, by Theorem 3.4.

Therefore
\begin{equation}
\eta_{\pi}^3 \subset \Gamma_{\pi}^3.
\end{equation}

Case 2: Now we show that $[\Gamma_{\pi}^3]^{\beta} \subset \Lambda_{\pi}^3$.

Let $y = \{y_{mnk}\}$ be an arbitrary point in $(\Gamma_{\pi}^3)^{\beta}$. If $y$ is not in $\Lambda_{\pi}^3$, then for each natural number $q$, we can find an index $m_qn_qk_q$ such that
THE GROWTH RATE OF $\Gamma^3$ DEFINED BY ORLICZ FUNCTION

Define $x = \{x_{mnk}\}$ by

$$\left(f\left(\frac{|y_{mnk}|}{\rho}\right)\right)_{P_{mnk}} > 1.$$  \hfill (3.10)

Consider the sequence $z = \{z_{mnk}\}$, where

$$f\left(\frac{x_{111}/\pi_{111}t_{111}}{\rho}\right) = f\left(\frac{x_{111}/\pi_{111}t_{111}}{\rho}\right) - s$$

with $s = \sum f\left(\frac{x_{mnk}}{\rho}\right)$; and

$$\left(f\left(\frac{z_{mnk}}{\rho}\right)\right) = \left(f\left(\frac{x_{mnk}}{\rho}\right)\right)_{m, n, k = 1, 2, 3, \cdots}$$

Then $z$ is a point of $\Gamma^3_{f\pi}$. Also $\sum f\left(\frac{z_{mnk}}{\rho}\right) = 0$. Hence $z$ is in $\Gamma^3_{f\pi}$.

But, by the equation (3.10), $\sum f\left(\frac{z_{mnk}}{\rho}\right)$ does not converge.

$\Rightarrow \sum x_{mnk}y_{mnk}$ diverges.

Thus the sequence $y$ would not be in $(\Gamma^3_{fM\pi})^\beta$. This contradiction proves that

$$\left(\Gamma^3_{fM\pi}\right)^{\beta} \subset \Lambda^3_{\pi}.$$  \hfill (3.11)

If we now choose $M = id$, where $id$ is the identity function and $y_{1nk}/\pi_{1nk}t_{1nk} = x_{1nk}/\pi_{1nk}t_{1nk} = 1$ and $y_{mnk}/\pi_{mnk}t_{mnk} = x_{mnk}/\pi_{mnk}t_{mnk} = 0$ for all $n, k$, then obviously $x \in \Gamma^3_{M\pi}$ and $y \in \Lambda^3_{\pi}$, but $\sum_{m,n,k=1}^{\infty} x_{mnk}y_{mnk} = \infty$, hence

$$y \notin \left(\Gamma^3_{M\pi}\right)^{\beta}.$$  \hfill (3.12)

From (3.11) and (3.12) we are granted

$$\left(\Gamma^3_{M\pi}\right)^{\beta} \subset \Lambda^3_{\pi}.$$  \hfill (3.13)

Hence (3.9) and (3.13) we are granted $\eta^3_{\pi} \subset \left[\Gamma^3_{M\pi}\right]^{\beta} \subset \Lambda^3_{\pi}$.

3.8. Proposition. The $\beta-$ dual space of $\Gamma^3_{f\pi}$ is $\Lambda^3_{f\pi}$

Proof: First, we observe that $\Gamma^3_{f\pi} \subset \Gamma^3_{f\pi}$, by Proposition 3.6. Therefore $\left(\Gamma^3_{f\pi}\right)^{\beta} \subset \left(\Gamma^3_{f\pi}\right)^{\beta}$.
But \( \Gamma_{f, \pi}^3 \subset \Lambda_{f, \pi}^3 \), by Proposition 3.7. Hence

\[
\Lambda_{f, \pi}^3 \subset (\Gamma_{f, \pi}^3)^\beta
\]

Next we show that \( (\Gamma_{f, \pi}^3)^\beta \subset \Lambda_{f, \pi}^3 \). Let \( y = (y_{mnk}) \in (\Gamma_{f, \pi}^3)^\beta \). Consider \( f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mnk} y_{mnk} \) with \( x = (x_{mnk}) \in \Gamma_{f, \pi}^3 \). Hence

\[
\left\{ \left( \frac{x_{mn}}{\pi_{mnk}} \right)^{1/m+n} \right\}
\]

is converges to zero.

Therefore

\[
[(\Im_{mn} - \Im_{mn+1} - \Im_{mn+2} - \Im_{m+1n} - \Im_{m+1n+1} - \Im_{m+1n+2} - \Im_{m+2n} - \Im_{m+2n+1} - \Im_{m+2n+2})] \in \Gamma_{f, \pi}^3.
\]

Hence

\[
d\left( [(\Im_{mn} - \Im_{mn+1} - \Im_{mn+2} - \Im_{m+1n} - \Im_{m+1n+1} - \Im_{m+1n+2} - \Im_{m+2n} - \Im_{m+2n+1} - \Im_{m+2n+2})] ; 0 \right) = 1.
\]

But

\[
|y_{mnk}| \leq \|f\|
\]

\[
d\left( [(\Im_{mn} - \Im_{mn+1} - \Im_{mn+2} - \Im_{m+1n} - \Im_{m+1n+1} - \Im_{m+1n+2} - \Im_{m+2n} - \Im_{m+2n+1} - \Im_{m+2n+2})] ; 0 \right) \leq \|f\| \cdot 1 < \infty \text{ for each } m, n, k. \text{ Thus } (y_{mnk}) \text{ is a triple growth rate of an bounded sequence and hence an growth rate of an triple analytic sequence. In other words } y \in \Lambda_{f, \pi}^3. \text{ But } y = (y_{mnk}) \text{ is arbitrary in } (\Gamma_{f, \pi}^3)^\beta. \text{ Therefore}
\]

\[
(\Gamma_{f, \pi}^3)^\beta \subset \Lambda_{f, \pi}^3
\]

From (3.14) and (3.15) we get \( (\Gamma_{f, \pi}^3)^\beta = \Lambda_{f, \pi}^3 \).

3.9. Proposition. \( \Gamma_{f, \pi}^3 \) has AK  

**Proof:** Let \( x = (x_{mnk}) \in \Gamma_{f, \pi}^3 \) and take the \( [m, n, k] \)th sectional sequence of \( x \). We have

\[
d(x, x^{[r,s,t]}) = \sup_{mnk} \left\{ \left( \frac{x_{mnk}}{\pi_{mnk}} \right)^{1/m+n+k} : m \geq r, n \geq s, k \geq t \right\} \to 0 \text{ as } [r, s, t] \to \infty.
\]

Therefore \( x^{[r,s,t]} \to x \) in \( \Gamma_{f, \pi}^3 \) as \( r, s, t \to \infty \). Thus \( \Gamma_{f, \pi}^3 \) has AK.

3.10. Proposition. \( \Gamma_{f, \pi}^3 \) is solid  

**Proof:** Let \( |x_{mnk}| \leq |y_{mnk}| \) and let \( y = (y_{mnk}) \in \Gamma_{f, \pi}^3 \). We have
\[
\left( \frac{x_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \leq \left( \frac{y_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \text{. But } \left( \frac{x_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \in \Gamma_{f_{\pi}}^3, \text{ because } y \in \Gamma_{f_{\pi}}^3. \text{ That is } \left( \frac{y_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \rightarrow 0 \Rightarrow \left( \frac{x_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty. \text{ Therefore } x = (x_{mnk}) \in \Gamma_{f_{\pi}}^3.
\]

3.11. Proposition. Let \( \Lambda - \) dual of \( \Gamma_{f_{\pi}}^3 \) is \( \Lambda_{f_{\pi}}^3 \)

\textbf{Proof:} Let \( y \in \Lambda - \) dual of \( \Gamma_{f_{\pi}}^3 \). Then \(|x_{mnk}y_{mnk}| \leq M^{m+n+k} \) for some constant \( M^{m+n+k} > 0 \) and for each \( x \in \Gamma_{f_{\pi}}^3 \). Therefore \(|y_{mnk}| \leq M^{m+n+k} \) for each \( m, n, k \) by taking
\[
x = (\exists_{mnk}) \text{ is a three dimensional matrix with } (\pi_{mnk} t_{mnk}) \text{ in the } (m, n, k)^{th} \text{ position and zero otherwise.}
\]
This shows that \( y \in \Lambda_{f_{\pi}}^3 \). Then
\[
(3.16) \quad (\Gamma_{f_{\pi}}^3)^{\Lambda} \subset \Lambda_{f_{\pi}}^3
\]

On the other hand, let \( y \in \Lambda_{f_{\pi}}^3 \). Let \( \epsilon > 0 \) be given. Then \(|y_{mnk}| < M^{m+n+k} \) for each \( m, n, k \) and for some constant \( M^{m+n+k} > 0 \). But \( x \in \Gamma_{f_{\pi}}^3 \). Hence \( \left( \frac{x_{mnk}}{\pi_{mnk} t_{mnk}} \right) < \epsilon^{m+n+k} \) for each \( m, n, k \) and for each \( \epsilon > 0 \). i.e \( |x_{mnk}| < \epsilon^{m+n+k} \pi_{mnk} t_{mnk} \). Hence
\[
|x_{mnk} y_{mnk}| = |x_{mnk}| |y_{mnk}| < \epsilon^{m+n+k} \pi_{mnk} t_{mnk} M^{m+n+k} = (\epsilon M)^{m+n+k} \pi_{mnk} t_{mnk}
\]
\[
\Rightarrow y \in (\Gamma_{f_{\pi}}^3)^{\Lambda}
\]
\[
(3.17) \quad \Lambda_{f_{\pi}}^3 \subset (\Gamma_{f_{\pi}}^3)^{\Lambda}
\]
From (3.16) and (3.17) we get \((\Gamma_{f_{\pi}}^3)^{\Lambda} = \Lambda_{f_{\pi}}^3 \).

3.12. Proposition. Let \((\Gamma_{f_{\pi}}^3)^{*}\) denote the dual space of \( \Gamma_{f_{\pi}}^3 \). Then we have \((\Gamma_{f_{\pi}}^3)^{*} = \Lambda_{f_{\pi}}^3 \).

\textbf{Proof:} We recall that \( x = (\exists_{mnk}) \) is a three dimensional matrix with \( (\pi_{mnk} t_{mnk}) \) in the \( (m, n, k)^{th} \) position and zero otherwise, with \( x = \left( \exists_{mnk}, \left\{ \left( \frac{x_{mnk}}{\pi_{mnk} t_{mnk}} \right)^{1/m+n+k} \right\} \right) \) is a three dimensional matrix with \( (1)^{1/m+n+k} \) in the \( (m, n, k)^{th} \) position and zero otherwise, which is a triple growth rate sequence of \( \Gamma_{f_{\pi}}^3 \). Hence \( \exists_{mnk} \in \Gamma_{f_{\pi}}^3 \). Let us take
\[ f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x_{mnk} y_{mnk} \] with \( x \in \Gamma_{f^\pi}^3 \) and \( f \in (\Gamma_{f^\pi}^3)^* \). Take \( x = (x_{mnk}) = \mathfrak{I}_{mnk} \in \Gamma_{f^\pi}^3 \). Then

\[ |y_{mnk}| \leq \|f\| d(\mathfrak{I}_{mnk}, 0) < \infty \] for each \( m, n, k \).

Thus \((y_{mnk})\) is a growth rate of bounded sequence and hence triple growth rate of an analytic sequence. In other words \( y \in \Lambda_{f^\pi}^3 \). Therefore \((\Gamma_{f^\pi}^3)^* = \Lambda_{f^\pi}^3 \).

**Competing Interests:** The authors declare that there is no conflict of interests regarding the publication of this research paper.

**References**


THE GROWTH RATE OF $\Gamma^3$ DEFINED BY ORLICZ FUNCTION


1 SQC and OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata, 700 108, West Bengal, India

2 Department of Mathematics, SASTRA University, Thanjavur-613 401, India

3 Department of Mathematics, National Institute of Technology, Silchar 788 010, District Cachar, Assam, India