Solving Integral Equations with a Wavelet Collocation Approach

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Abstract

In this article we describe how to approximate the solution of an integral equation using a wavelet basis. The same idea of constructing an approximation function with a wavelet basis we examined by solving ODEs und PDEs. In the example we use the Shannon wavelet.

INTRODUCTION

In the wavelet theory a scaling function ϕ is used, which has properties that are defined in the MSA (multi scale analysis). Through the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a sequence of subspaces with the following property:

$$... \subset V_{-1} \subset V_0 \subset V_1 \subset ... \subset L^2(\mathbb{R}),$$

$\{\phi_{j,k}(t)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

We use the following approximation function to approximate the solution of integral equations (EQ):

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t) \quad .$$

k_{max} and k_{min} depend on the approximation interval $[t_0, t_{end}]$.

The idea is to replace the unknown function in an integral equation trough the approximation function y_j. Then we have $m := |k_{max} - k_{min}| + 1$ unknown coefficients c_k. The same idea is used for solving ODEs with the collocation method. Then we construct a system of equations for the unknown coefficients, where the integral equation with the substituted approximation function should be fulfilled at m discrete (for example equidistant) points in the integration area of the integral equation.

If we have for example an initial value problem

$$y' = f(y,t)$$
$$y(t_0) = y_0$$

we can write it as an integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau),\tau)\,d\tau$$

If we want to approximate the solution y on the interval $I = [t_0, t_{end}]$.

We use the collocation points t_i, with $t_i = t_0 + i \cdot h$ (i = 1, 2, .., m) and

$$h = \frac{t_{end} - t_0}{m} \ .$$

Then we can solve the following equations for the c_k:

$$y(t_i) = y_0 + \int_{t_0}^{t_i} f(y(\tau), \tau) d\tau \quad \text{, with } i = 1, 2, .., m.$$

If f is linear in y we get a linear system of equations, for example $f(y(t), t) = h(t) \cdot y(t)$, we get

$$y(t_i) = y_0 + \int_{t_0}^{t_i} h(\tau) \cdot \sum_{k=k_{min}}^{k_{max}} c_k \cdot \phi_{j,k}(\tau) d\tau \text{ , with } i = 1, 2, .., m$$

and so:

$$y(t_i) = y_0 + \sum_{k=k_{min}}^{k_{max}} c_k \cdot \int_{t_0}^{t_i} h(\tau) \cdot \phi_{j,k}(\tau) d\tau \text{ , with } i = 1, 2, .., m.$$

Analogues we can solve numerically other types of integral equations. Now we solve approximately an integral equation in the following example.

Example:

The integral equation is

(1) $$y(t) = t + \int_0^t y(\tau) \cdot sin(t - \tau) d\tau$$

with the solution $$y(t) = t + \frac{1}{6} t^3 \ .$$

Here we set (with Shannon's ϕ) (with $k_{max} = - k_{min} = 10$ und $j = 1$)

$$y_1(t) = \sum_{k=-10}^{10} c_k \cdot \phi_{1,k}(t)$$

in the integral equation (1) and we solved the equations

$$y_1(x) = x + \int_0^x y_1(t) \cdot \sin(x - t) dt$$

with $x = 0, \frac{1}{5}, \frac{2}{5}, ..., 4$ for the coefficients $c_{-10}, ..., c_{10}$ with numerical evaluation of the integrals.

Here is the graph of y_1:

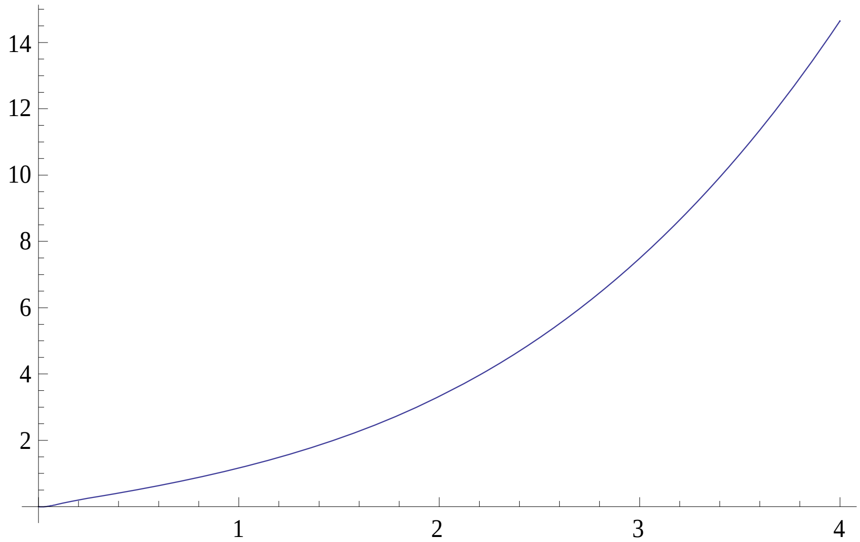

Fig. 1

And the graph of $y_1 - y$:

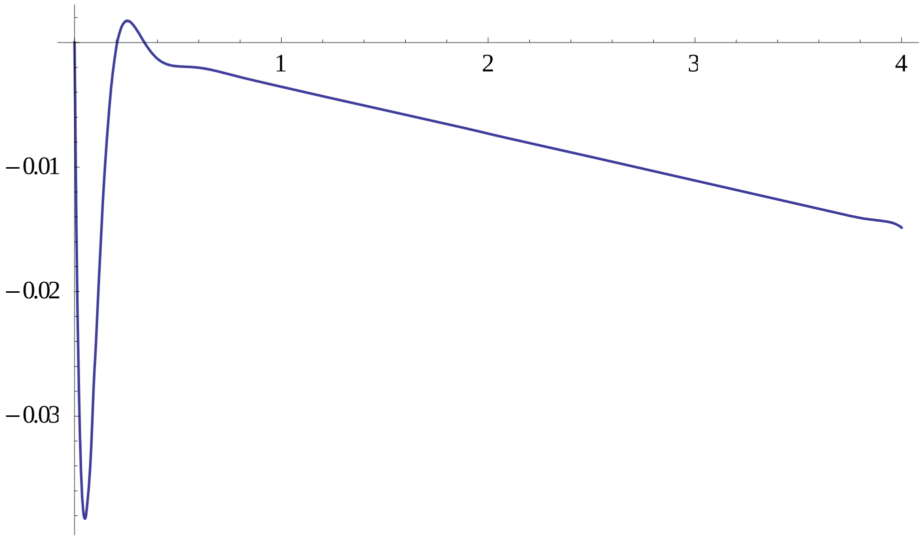

Fig. 2

Now we solve the following minimum problem instead of the equations above and use :

$$\min_{c_k} Q(\bar{c}) = \min_{c_k} \sum_{i=0}^{40} \Big(-y_1(i/10) + i/10 + \int_0^{i/10} y_1(t)\cdot \sin(i/10 - t)\,dt\Big)^2$$

$$= \min_{c_k} \sum_{i=0}^{40} \Big(-y_1(i/10) + i/10 + \sum_{k=-10}^{10} c_k \int_0^{i/10} \phi_{1,k}(t)\cdot \sin(i/10 - t)\,dt\Big)^2 \quad .$$

Here we can use more collocation points and we got better results. We now see the curve of the difference $y_1 - y$:

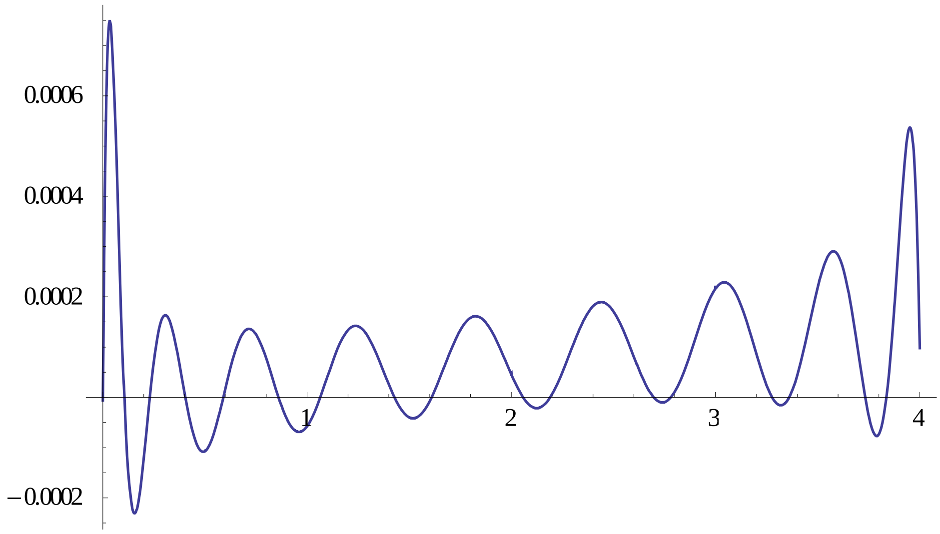

Fig. 3

With k_{max} = - k_{min} = 15 we get better results, too. Here is the graph of $y_1 - y$ in that case:

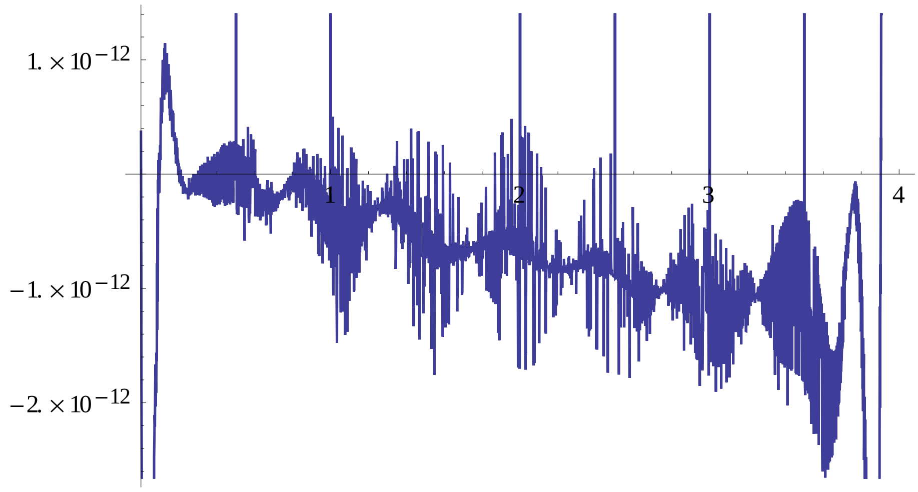

Fig. 4

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