Theorem 7 *[31] Let A defined as above. With Laplace transform $F(s)$, then the Elzaki transform $T(v)$ of $f(t)$ is given by :*

$$T(v) = vF(\frac{1}{v}).$$

Theorem 8 *Suppose $T(v)$ is the Elzaki transform of the function $f(t)$ then*

$$E\{\ ({}^{c}D^{\alpha}_{0+}f)(t), v\} = \frac{T(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0). \tag{11}$$

Proof (see [32]).

3 Modified fractional homotopy analysis transform method (MFHATM)

Kangle Wang and Sanyang Liu [15] gives the idea of the basis of this method. They consider the following general time-fractional differential equation with the initial condition as :

$$\begin{array}{c} {}^{c}D^{n\alpha}_{t}U(x,t) + LU(x,t) + RU(x,t) = g(x,t), \\ n-1 < n\alpha \leqslant n, \\ U(x,0) = h(x), \end{array} \tag{12}$$

where ${}^{c}D^{n\alpha}_{t}$ is the Caputo fractional derivative operator, ${}^{c}D^{n\alpha}_{t} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, L is the linear operator, R is the general nonlinear operator and $g(x,t)$ is a continuous functions.

Applying Elzaki transform on both sides of Eq.(12), we can get :

$$E\left[\ {}^{c}D^{n\alpha}_{t}U(x,t)\right] + E\left[LU(x,t) + RU(x,t) - g(x,t)\right] = 0, \tag{13}$$

Using the property of Elzaki transform, we have the following form:

$$E\left[\ U(x,t)\right] - v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}U^{(k)}(x,0) + v^{n\alpha}E\left[LU(x,t) + RU(x,t) - g(x,t)\right] = 0, \tag{14}$$

Define the nonlinear operator :

$$\begin{aligned} N[\phi(x,t;p)] &= E\,[\;\phi(x,t;p)] - v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k} h^{(k)}(x,0) \\ &\quad + v^{n\alpha} E\,[L\phi(x,t;p) + R\phi(x,t;p) - g(x,t;p)] \end{aligned} \tag{15}$$

By means of homotopy analysis method [1], we construct the so-called the zero-order deformation equation :

$$(1-q)E[\phi(x,t;p) - \phi(x,t;0) = phH(x,t)N[\phi(x,t;p)], \tag{16}$$

where p is an embedding parameter and $p \in [0, 1]$, $H(x;t) \neq 0$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, E is an auxiliary linear Elzaki operator. When $p = 0$ and $p = 1$, we have :

$$\begin{cases} \phi(x,t;0) = u_0(x,t), \\ \phi(x,t;1) = u(x,t). \end{cases} \tag{17}$$

When P increases from 0 to 1, the $\phi(x,t,p)$ various from $U_0(x,t)$ to $U(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have :

$$\phi(x,t;p) = U_0(x,t) + \sum_{m=1}^{+\infty} U_m(x,t)p^m, \tag{18}$$

where

$$U_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\,|_{p=0}\,. \tag{19}$$

When $p = 1$, the (18) becomes :

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{+\infty} U_m(x,t). \tag{20}$$

Define the vectors :

$$\overrightarrow{U}_n = \{U_0(x,t), U_1(x,t), U_2(x,t), \ldots, U_n(x,t)\}. \tag{21}$$

Differentiating (16) $m-$times with respect to p, then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called mth order deformation equation :

$$E[U_m(x,t) - \chi_m U_{m-1}(x,t)] = hpH(x,t)\Re_m(\overrightarrow{U}_{m-1}(x,t)), \tag{22}$$

where

$$\Re_m(\overrightarrow{U}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N(x,t;p)}{\partial p^{m-1}}\mid_{p=0}, \tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

Applying the inverse Elzaki transform on both sides of Eq.(22), we can obtain :

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + E^{-1}\left[hpH(x,t)\Re_m(\overrightarrow{U}_{m-1}(x,t))\right]. \tag{24}$$

The mth deformation equation (24) is a linear which can be easily solved. So, the solution of Eq.(12) can be written into the following form :

$$U(x,t) = \sum_{m=0}^{N} U_m(x,t), \tag{25}$$

when $N \to \infty$, we can obtain an accurate approximation solution of Eq.(12).

The proof of the convergence of the modified fractional homotopy analysis transform method (MFHATM) (see [2]).

4 Application of the MFHATM Method

In this section, we apply the modified fractional homotopy analysis transform method (MFHATM) for solving the following nonlinear porous medium equation with time-fractional derivative (1) in the two cases : $k = -1$ and $k = -4/3$.

4.1 Example

First, we take $k = -1$ in equation (1)[24], we get :

$$^{c}D_t^{\alpha} u = \left(u^{-1} u_x\right)_x, \quad 0 < \alpha \leqslant 1, \tag{26}$$

with the initial condition :

$$u(x, 0) = \frac{1}{x}. \tag{27}$$

Applying Elzaki transform on both sides of Eq.(26), we can get :

$$E\left[u\right] - v^2 u(x, 0) = v^{\alpha} E\left[\left(u^{-1} u_x\right)_x\right] \tag{28}$$

From (28) and the initial condition (27), we have :

$$E\left[u\right] - v^2 \frac{1}{x} - v^{\alpha} E\left[\left(u^{-1} u_x\right)_x\right] = 0. \tag{29}$$

We take the nonlinear part as :

$$N[\phi(x, t, p)] = E\left[\phi\right] - v^2 \frac{1}{x} - v^{\alpha} E\left[\left(\phi^{-1} \phi_x\right)_x\right]. \tag{30}$$

We construct the so-called the zero-order deformation equation with assumption $H(x; t) = 1$, we have :

$$(1 - q)E[\phi(x, t; p) - \phi(x, t; 0) = phN[\phi(x, t; p)]. \tag{31}$$

When $p = 0$ and $p = 1$, we can obtain :

$$\begin{cases} \phi(x, t; 0) = u_0(x, t), \\ \phi(x, t; 1) = u(x, t). \end{cases}$$

Therefore, we have the mth order deformation equation :

$$E[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h\Re_m(\overrightarrow{u}_{m-1}(x, t)). \tag{32}$$

Operating the inverse Elzaki operator on both sides of Eq.(32), we get :

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + E^{-1}[h\Re_m(\overrightarrow{u}_{m-1}(x, t))]. \tag{33}$$

From Eq.(33), we have :

$$\begin{aligned} u_1(x,t) &= hE^{-1}[\,\Re_1(\overrightarrow{u}_0(x,t))], \\ u_2(x,t) &= u_1 + hE^{-1}[\,\Re_2(\overrightarrow{u}_1(x,t))], \\ u_3(x,t) &= u_2 + hE^{-1}[\,\Re_3(\overrightarrow{u}_2(x,t))], \\ &\vdots \end{aligned} \tag{34}$$

where

$$\begin{aligned} \Re_1(\overrightarrow{u}_0(x,t)) &= E\left[u_0\right] - v^2\frac{1}{x} - v^\alpha E\left[\left(u_0^{-1}u_{0x}\right)_x\right], \\ \Re_2(\overrightarrow{u}_1(x,t)) &= E\left[u_1\right] - v^\alpha E\left[\left(u_0^{-1}u_{1x} - u_0^{-2}u_1u_{0x}\right)_x\right], \\ \Re_3(\overrightarrow{u}_2(x,t)) &= E\left[u_2\right] - v^\alpha E\left[\left(u_0^{-3}u_1^2u_{0x} - u_0^{-2}u_2u_{0x} - u_0^{-2}u_1u_{1x} + u_0^{-1}u_{2x}\right)_x\right], \\ &\vdots \end{aligned} \tag{35}$$

Using the initial condition (27), the iteration formulas (34) and (35), we obtain :

$$\begin{aligned} u_0(x,t) &= \frac{1}{x}, \\ u_1(x,t) &= -\frac{h}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= -\frac{h}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{h^2}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} + 2\frac{h^2}{x^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3(x,t) &= -\frac{h}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{h^2}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\frac{h^2}{x^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad -\frac{h^2}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{h^3}{x^2}\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\frac{h^3}{x^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad +3\frac{h^3}{x^4}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 12\frac{h^3}{x^4}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\vdots \end{aligned} \tag{36}$$

Thus, we use four terms in evaluating the approximate solution :

$$u(x,t) = \sum_{m=0}^{3} u_m(x,t).$$

When $h = -1$, the approximate solution of Eq.(26), is given by :

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{3} u_m(x,t) \\ &= \frac{1}{x} + \frac{1}{\Gamma(\alpha+1)}\frac{t^{\alpha}}{x^2} + \frac{2}{\Gamma(2\alpha+1)}\frac{t^{2\alpha}}{x^3} - \frac{3\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\frac{t^{3\alpha}}{x^4} \\ &\quad + \frac{12}{\Gamma(3\alpha+1)}\frac{t^{3\alpha}}{x^4}. \end{aligned}$$

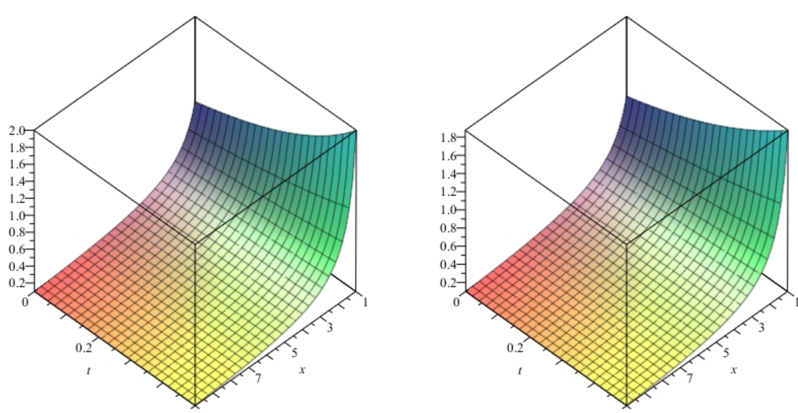
If $\alpha = 1$, we obtain :

$$u(x,t) = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + \cdots$$

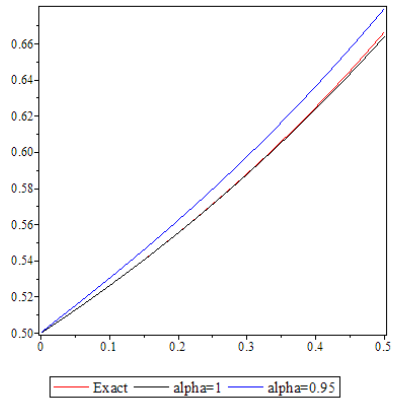
That gives :

$$u(x,t) = \frac{1}{x-t}, \quad \left|\frac{t}{x}\right| < 1, \;\; x \neq 0,$$

which is an exact solution to the porous medium equation as presented in [24].



(a) (b)



(c)

Fig. 1 : (a) Exact solution, (b) the approximate solution in the case $\alpha = 1$, (c) The exact solution and approximate solutions to the Eq.(26) for different values of α when $x = 2$.

4.2 Example

Second, we take $k = -4/3$ in equation (1)[24], we get :

$$^{c}D_t^{\alpha} u = \left(u^{-4/3} u_x\right)_x, \quad 0 < \alpha \leqslant 1, \tag{37}$$

with the initial condition :

$$u(x, 0) = (2x)^{-3/4}. \tag{38}$$

Applying Elzaki transform on both sides of Eq.(37), we get :

$$E\left[u\right] - v^2 u(x, 0) = v^{\alpha} E\left[\left(u^{-4/3} u_x\right)_x\right] \tag{39}$$

From (39) and the initial condition (38), we have :

$$E\left[u\right] - v^2 (2x)^{-3/4} - v^{\alpha} E\left[\left(u^{-4/3} u_x\right)_x\right] = 0.$$

We take the nonlinear part as :

$$N[\phi(x, t, p)] = E\left[\phi\right] - v^2 (2x)^{-3/4} - v^{\alpha} E\left[\left(\phi^{-4/3} \phi_x\right)_x\right].$$

We construct the so-called the zero-order deformation equation with assumption $H(x;t)=1$, we have :

$$(1-q)E[\phi(x,t;p)-\phi(x,t;0)=phN[\phi(x,t;p)].$$

When $p=0$ and $p=1$, we can obtain :

$$\begin{cases} \phi(x,t;0)=u_0(x,t), \\ \phi(x,t;1)=u(x,t). \end{cases}$$

Therefore, we have the mth order deformation equation :

$$E[u_m(x,t)-\chi_m u_{m-1}(x,t)]=h\Re_m(\overrightarrow{u}_{m-1}(x,t)). \tag{40}$$

Operating the inverse Elzaki operator on both sides of Eq.(40), we get :

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+E^{-1}[h\Re_m(\overrightarrow{u}_{m-1}(x,t))], \tag{41}$$

From Eq.(41), we have :

$$\begin{aligned} u_1(x,t) &= hE^{-1}[\ \Re_1(\overrightarrow{u}_0(x,t))], \\ u_2(x,t) &= u_1+hE^{-1}[\ \Re_2(\overrightarrow{u}_1(x,t))], \\ u_3(x,t) &= u_2+hE^{-1}[\ \Re_3(\overrightarrow{u}_2(x,t))], \\ &\ \vdots \end{aligned} \tag{42}$$

where

$$\begin{aligned} \Re_1(\overrightarrow{u}_0(x,t)) &= E[u_0]-v^2(2x)^{-3/4}-v^\alpha E\left[\left(u_0^{-4/3}u_{0x}\right)_x\right], \\ \Re_2(\overrightarrow{u}_1(x,t)) &= E[u_1]-v^\alpha E\left[\left(-\frac{4}{3}u_0^{-7/3}u_1u_{0x}+u_0^{-4/3}u_{1x}\right)_x\right], \\ \Re_3(\overrightarrow{u}_2(x,t)) &= E[u_2]-v^\alpha E\left[\left(\begin{array}{c} \frac{14}{9}u_0^{-10/3}u_1^2u_{0x}-\frac{4}{3}u_0^{-7/3}u_2u_{0x} \\ -\frac{4}{3}u_0^{-7/3}u_1u_{1x}+u_0^{-4/3}u_{2x} \end{array}\right)_x\right], \\ &\ \vdots \end{aligned} \tag{43}$$

Using the initial condition (38), the iteration formulas (42) and (43), we obtain :

$$\begin{aligned}
u_0(x,t) &= (2x)^{-3/4},\\
u_1(x,t) &= -\frac{9}{4}h\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2(x,t) &= -\frac{9}{4}h\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{9}{4}h^2\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{189}{16}h^2\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_3(x,t) &= -\frac{9}{4}h\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{9}{4}h^2\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{189}{16}h^2\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad -\frac{9}{4}h^2\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{9}{4}h^3\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{189}{16}h^3\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad +\frac{1089}{32}h^3\,(2x)^{-15/4}\,\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{189}{16}h^2\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad +\frac{189}{16}h^3\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{14553}{64}h^3\,(2x)^{-15/4}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad +\frac{2079}{16}h^2\,(2x)^{-15/4}\,\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.\\
&\quad \vdots
\end{aligned}$$

Thus, we use four terms in evaluating the approximate solution :

$$u(x,t)=\sum_{m=0}^{3}u_m(x,t).$$

When $h=-1$, the approximate solution of Eq.(37), is given by :

$$\begin{aligned}
u(x,t) &= \sum_{m=0}^{3}u_m(x,t)\\
&= (2x)^{-3/4}+\frac{9}{4}\,(2x)^{-7/4}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{189}{16}\,(2x)^{-11/4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad +\frac{3069}{32}\,(2x)^{-15/4}\,\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{14553}{64}\,(2x)^{-15/4}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.
\end{aligned}$$

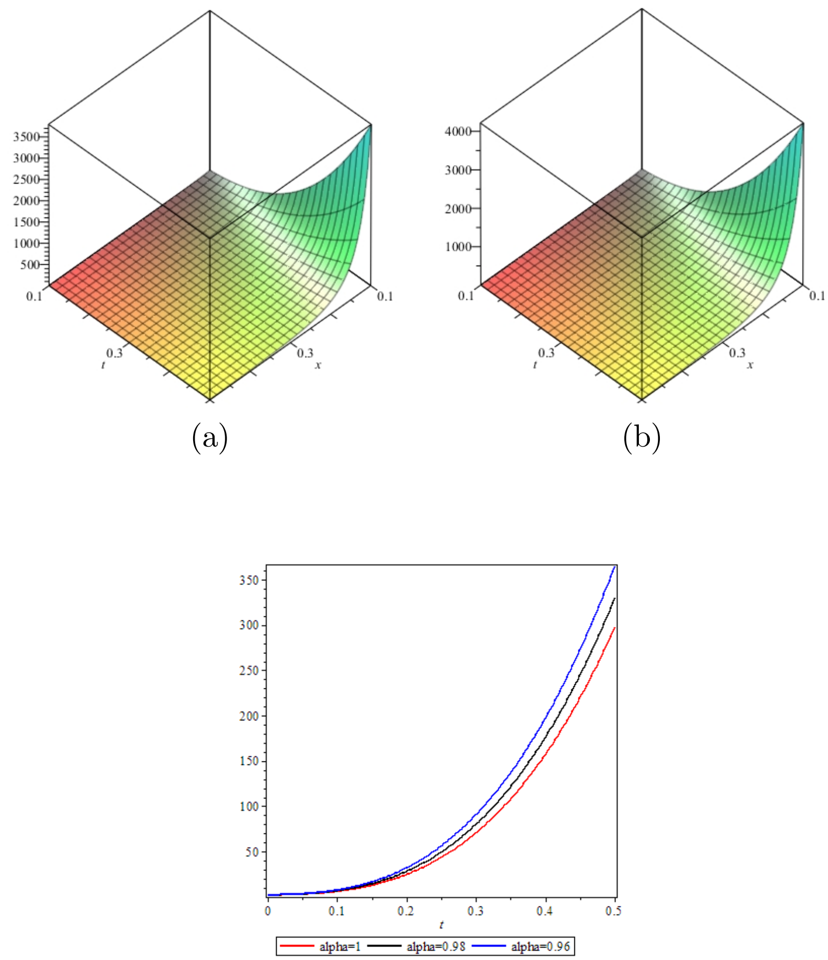
If $\alpha=1$, we obtain :

$$u(x,t)=(2x)^{-3/4}+\frac{9}{4}\,(2x)^{-7/4}\,t+\frac{189}{32}\,(2x)^{-11/4}\,t^2+\frac{8943}{128}\,(2x)^{-15/4}+\cdots,$$

or

$$u(x,t) = 2^{-3/4} \times x^{-3/4} + (9 \times 2^{-15/4} \times x^{-7/4})t + (189 \times 2^{-31/4} \times x^{-11/4})t^2 + (8943 \times 2^{-43/4} \times x^{-15/4})t^3 + \cdots ,$$

which is the same approximate solution to the porous medium equation as presented in [24].



(a) (b)

(c)

Fig. 2 : (a) The approximate solution when $\alpha = 1$, (b) The approximate solution when $\alpha = 0.98$, (c) The approximate solutions to the Eq.(37) for different values of α when $x = 0.2$.

Remark 9 *For graph approximate solutions, we took only four terms.*

5 Conclusion

The coupling of homotopy analysis method (HAM) and Elzaki transform method proved very effective to solve nonlinear partial differential equations. The modified fractional homotopy analysis transform method (MFHATM), is suitable for such problems and is very user friendly. From the obtained results, it is clear that the MFHATM yields very accurate, exact and approximate solutions using only a few iterates. As a result, the conclusion that comes through this work is that MFHATM can be applied to other nonlinear fractional partial differential equations of higher order, due to the efficiency and flexibility in the application to get the possible results.

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