Fourier Properties of Approximations with Functions on a Compact Interval using Daubechies Wavelets

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Abstract

One way to approximate a non $L^2(R)$ function g on a compact interval I is to use y = $1_I \cdot g$ (with the indicator function 1_I). Here we calculate the orthogonal projection from $1_I \cdot g$ to V_j. With the factor 1_I we cut the function g and hence we generally get jumps on the edges of the interval I. This leads to a bad decay behavior of the base coefficients. In this paper we show that the orthogonal projection from $1_I \cdot g$ to Vj with Daubechies wavelet leads to a Fourier series and with a small j we get a too fast decay behavior of the coefficients of the approximation function y_j in comparison to y, which leads to all well known problems with Fourier approximations when we do not consider higher frequencies (e.g. oscillation effects).

Introduction

In the wavelet theory a scaling function ϕ is used, which belongs to a MSA (multi scale analysis). From the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a the sequence of subspaces with the following property:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k\epsilon Z}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

If we want to get an approximation of a function g on a compact interval I then g does not need be in $L^2(R)$ but must be in $L^2(I)$ if we use the function $1_I \cdot g$. But this type of function can lead to bad approximations what we will see soon. Generally it would be better to use a continuous or even better continuously differentiable extension of $1_I \cdot g$ on R, which also is quadratic integrable on R (see [6]).

We get an approximation function (as a orthogonal projection form y on V_j)

$$y_j(t) := \sum_k c_k \cdot \phi_{j,k}(t)$$

with

$$c_k = \langle y, \phi_{j,k} \rangle = \int_{-\infty}^{\infty} y(t) \cdot \overline{\phi_{j,k}(t)} dt \ .$$

Example:
We use the Daubechies wavelet of order m = 4 and calculate the orthogonal projection of $y(t) = e^{-t} \cdot 1_{[0,1]}(t)$ on V_2. For many functions without a jump we would get with j = 2 a very good approximation, but not here. Because of the compact support of the scaling function and the function y we only need for y_j a finite summation area for k = -6, -5, ..., 3.

Here we see thee graph of y with its approximation y_j:

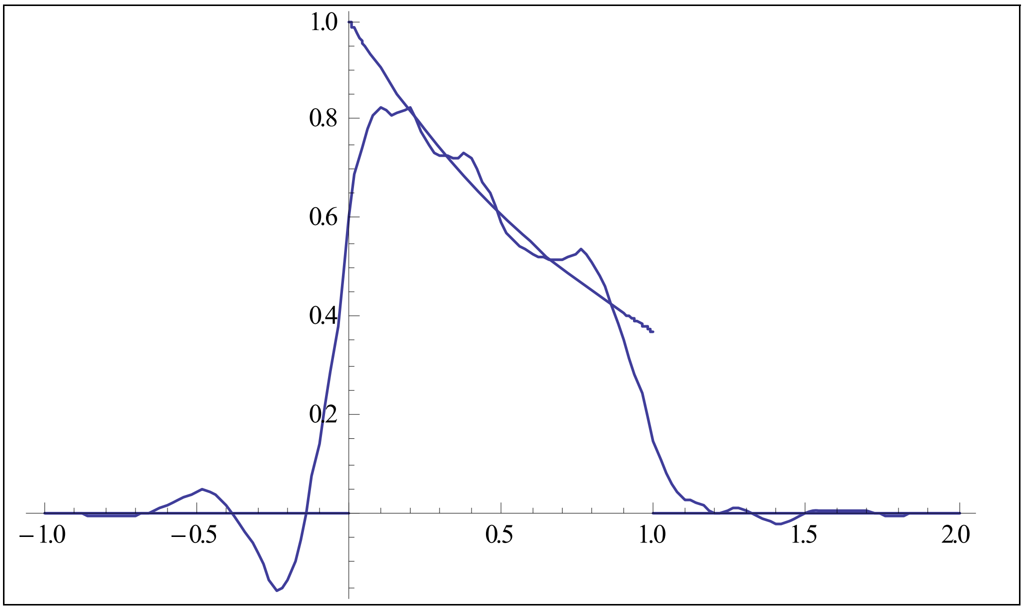

Figure 1

In [6] we described the same effect if we use the Shannon wavelet and we showed that the approximation function y_j by using the Shannon wavelet is almost a partial sum of a Fourier series with coefficients that have a bad decay behaviour. The Scaling function of the Shannon wavelet has a Fourier transform which is a rectangle function. The amplitude spectrum of the Fourier transform of the scaling function of the Daubechies wavelet of order 4 looks similar as a rectangle function (like all scaling functions of the Daubechies wavelet if the order m is not too small). It has no compact support but the decay is relative fast. That's one suggestion, but we well soon see, that even the best approximation of y in V_j is a Fourier series which has coefficients with a too fast decay behaviour in comparison to the Fourier series of the function y, so that we get a bad approximation if j is not very big.

Here is the amplitude spectrum of $\Phi^{D}_{2,k}$:

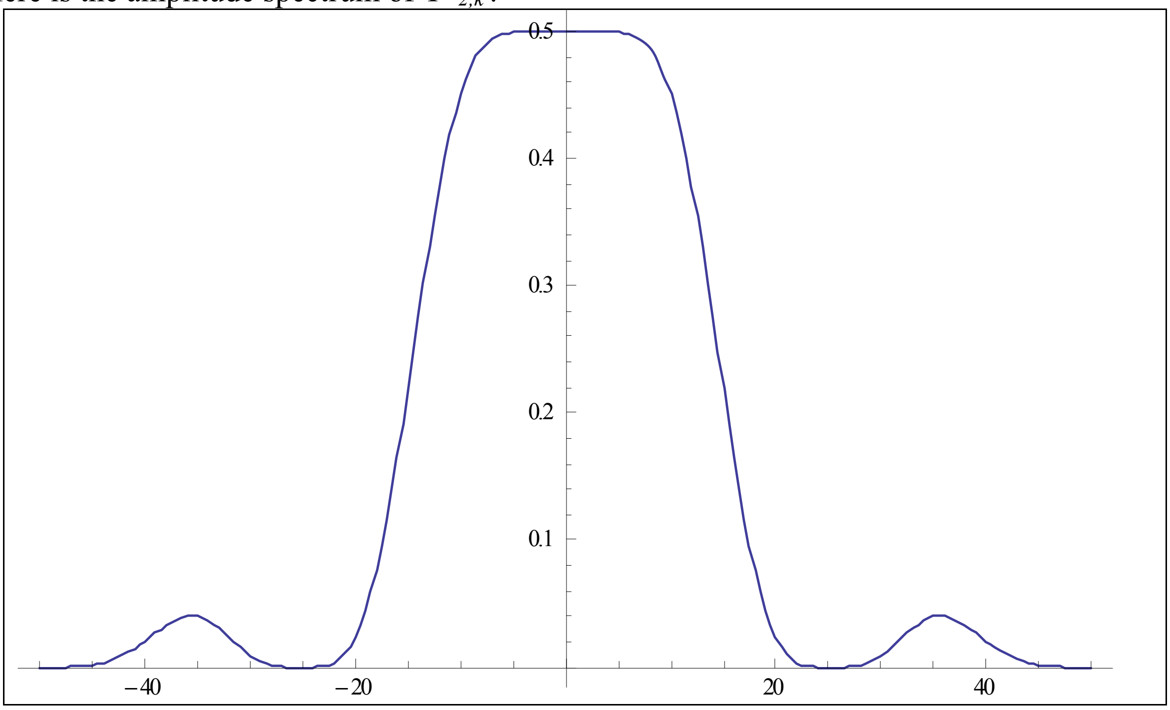

Figure 2

y and y_j and their Fourier Series

For the approximation of $y = 1_I \cdot g$ (with $I = [a, b]$) with a Daubechies wavelet of order m only a finite number of the coefficients c_k are unequal to zero because the scaling function $\phi^D{}_{j,k}$ of that Daubechies wavelet has compact support and $y = 1_I \cdot g$ too:

$$c_k = \left\langle f, \phi^D{}_{j,k} \right\rangle = \int_{-\infty}^{\infty} 1_I \cdot y(t) \cdot \overline{\phi^D{}_{j,k}(t)} dt = \int_a^b g(t) \cdot \overline{\phi^D{}_{j,k}(t)} dt$$

$$supp\ \phi^D = [0, 2m-1] \Rightarrow supp\ \phi^D{}_{j,k} = [2^{-j}k, 2^{-j}(2m-1+k)]$$

$c_k = 0$ for all k with $[2^{-j}k, 2^{-j}(2m-1+k)] \cap [a, b] = \{\}$ (or general only $\{x\}$ with a real x).

We assume that the biggest k with $c_k \neq 0$ is k_{max} and the smallest k is k_{min}.

So the best approximation or orthogonal projection of y on V_j is:

$$y_j(t) := \sum_{k=k_{min}}^{k_{max}} c_k \cdot \phi^D{}_{j,k}(t)$$

The base elements of V_j which are not equal to zero in $[a, b]$ are in a subspace of V_j:

That subspace is $span\{\phi^D{}_{j,k}\}_{k \in M}$ with $M = \{k_{min}, k_{min} + 1, \ldots, k_{max}\}$ (or generally its closed hull). If we choose a smallest integer $\tilde{j}$, so that

$$\bigcup_{k \in M} supp\ \phi^D{}_{j,k} \subseteq [-2^{\tilde{j}} \cdot \pi, 2^{\tilde{j}} \cdot \pi]$$

then in the Fourier space all the $\{\phi^D{}_{j,k}\}_{k \in M}$ can be written as a Shannon series (with the base elements $\phi^S{}_{j,\tilde{k}}$ from the scaling function of the Shannon wavelet) and in the original space the Shannon series are Fourier series, so

$$\phi_{j,k}(t) = \frac{2^{-\tilde{j}/2}}{2\pi} \cdot 1_{[-2^{\tilde{j}} \cdot \pi,\, 2^{\tilde{j}} \cdot \pi]}(t) \cdot \sum_{\tilde{k}} b_{\tilde{k}}^k \cdot e^{-i \cdot t \cdot \tilde{k}/2^{\tilde{j}}} \qquad (1)$$

for almost all t (compare with [6]). If Φ^D is the Fourier transform of ϕ^D and $\Phi^D{}_{j,k}$ is the Fourier transform of $\phi^D{}_{j,k}$, we get the coefficients $b_{\tilde{k}}^k$ through

$$b_{\tilde{k}}^k = 2^{-\tilde{j}/2} \cdot \Phi^D{}_{j,k}(2^{-\tilde{j}} \cdot \tilde{k})$$

and therefore through:

$$b_{\tilde{k}}^{k} = \underbrace{2^{-\tilde{j}/2+j/2} \cdot \Phi^{D}(2^{-(j+\tilde{j})} \cdot \tilde{k})}_{=:B(\tilde{k})} \cdot e^{i \cdot 2^{-(\tilde{j}+j)} \cdot \tilde{k} \cdot k} \qquad (2)$$

Analogous we get the representation of *y:*

$$y(t) = \frac{2^{-\tilde{j}/2}}{2\pi} \cdot 1_{[-2^{\tilde{j}} \cdot \pi,\, 2^{\tilde{j}} \cdot \pi]}(t) \cdot \sum_{r} a_r \cdot e^{-i \cdot t \cdot r / 2^{\tilde{j}}} \qquad (3)$$

with $a_r = 2^{\tilde{j}/2} \cdot Y(2^{-\tilde{j}} \cdot r)$.

So y can be written as a Fourier series but the coefficients a_r have the order $O(1/|r|)$ if y has jumps at the edges of I as in the example above.

So we would get for the coefficients c_k:

$$c_k = \frac{1}{2\pi} \sum_{r} a_r \cdot \overline{b_r^k} \qquad (4)$$

For easier notation we leave out the factor $1_{[-2^{\tilde{j}} \cdot \pi,\, 2^{\tilde{j}} \cdot \pi]}(t)$ and we consider that $t \in [-2^{\tilde{j}} \cdot \pi, 2^{\tilde{j}} \cdot \pi]$. No we get with (1):

$$y_j(t) = \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi^{D}{}_{j,k}(t)$$

$$= \frac{2^{-\tilde{j}/2}}{2\pi} \cdot \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \sum_{\tilde{k}} b_{\tilde{k}}^{k} \cdot e^{-i \cdot t \cdot \tilde{k} / 2^{\tilde{j}}}$$

$$= \frac{2^{-\tilde{j}/2}}{2\pi} \cdot \sum_{\tilde{k}} e^{-i \cdot t \cdot \tilde{k} / 2^{\tilde{j}}} \cdot \underbrace{\sum_{k=k_{\min}}^{k_{\max}} c_k \cdot b_{\tilde{k}}^{k}}_{:=a_{\tilde{k}}^{approx}}$$

$$= \frac{2^{-\tilde{j}/2}}{2\pi} \cdot \sum_{\tilde{k}} a_{\tilde{k}}^{approx} \cdot e^{-i \cdot t \cdot \tilde{k} / 2^{\tilde{j}}}$$

So we see that y_j can be written as a Fourier series, but in y the coefficients $a_{\tilde{k}}^{approx}$ have in comparison to the coefficients $a_{\tilde{k}}$ a too fast decay behaviour. So with $a_{\tilde{k}}^{approx}$ we do not consider the 'higher frequencies'. That is what we see if we take a look at $a_{\tilde{k}}^{approx}$. With (2) we get:

$$a_{\tilde{k}}^{approx} = \sum_{k=k_{min}}^{k_{max}} c_k \cdot b_{\tilde{k}}^{k} = B(\tilde{k}) \cdot \sum_{k \in M} c_k \cdot e^{i \cdot 2^{-(\tilde{j}+j)} \cdot \tilde{k} \cdot k} \qquad (5)$$

$B(\tilde{k})$ is calculated with the function values $\Phi^D(2^{-(j+\tilde{j})} \cdot \tilde{k})$. Φ^D is the Fourier transform of the scaling function of order *m* from the Daubechies wavelet. Φ^D has (with a not too small *m*) a very strong decay behaviour and for a small *j* the factor $B(\tilde{k})$ does too. So the needed higher frequencies are not considered.

Using (4), (5) and (2) we can write the coefficients $a_{\tilde{k}}^{approx}$ with a_r :

$$a_{\tilde{k}}^{approx} = \frac{1}{2\pi} B(\tilde{k}) \cdot \sum_r a_r \cdot \overline{B(r)} \cdot \sum_{k \in M} c_k \cdot e^{i \cdot 2^{-(\tilde{j}+j)} \cdot (\tilde{k}-r) \cdot k}$$

Conclusion

We saw, that we can write the approximation function from the Daubechies wavelets as a fourier series, if the function has compact support. If the function has jumps like generally functions of the type $1_I \cdot g$, then y_j is for small *j* a bad approximation because the decay of the fourier coefficients is too fast and so we get oscillation effects.

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