# *VARIATION ITERATION ADOMIAN DECOMPOSITION METHOD FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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## Abstract

In this paper, we present a numerical method called the Variation iteration Adomian decomposition method (VIADM) for solving nonlinear partial differential equations (PDEs). The method modifies the traditional formulation of the variation iteration decomposition method (VIDM) such that it converges more rapidly to the analytic solution. Also, the present method is explicit as it requires no special treatment of linearization, perturbation or discretization. Two examples are considered for experimentation. The resulting numerical evidences show that the present method is effective, efficient and reliable as compared with VIDM and homotopy perturbation method (HPM) available in the literature.

**Keywords:** Linear and nonlinear partial differential equations; Adomian decomposition method; Lagrange multiplier; Variation iteration method

## Introduction

Nonlinear partial differential equations are not new in the field of mathematics as it serves as a tool in the stimulation of real life situations. The solution to this class of equations is not easy to achieve. Hence, a lot of researches have been done by various researchers to solve this class of equations. Moreover, a lot of research is still ongoing with the sole aim of designing more effective and efficient mathematical algorithms for solving this class of equations. Known conventional analytic methods are often insufficient in handling this class of problems. To this effect, numerical methods have become relevant in solving this class of equations. Some popular numerical methods available include the Adomian decomposition method [1-4], the homotopy perturbation method [5-7], the variation iteration method [8-9], etc. In recent times, new numerical methods has also emerged, which include, the Elzaki transform decomposition method [10], the Adomian decomposition method coupled with Sumudu transform method [11], the Adomian decomposition method coupled with Laplace transform method [12], the variation iteration method coupled with Sumudu transform method [13], etc.

The basic motivation behind the present study is the development of a new numerical method called the variation iteration Adomian decomposition method (VIADM) for solving nonlinear partial differential equations. The method modifies the traditional formulation of the variation iteration decomposition method (VIDM) such that it converges more rapidly to the analytic solution. In the case of variation iteration decomposition method (VIDM), the components $u_n(x,t)$, $n \geq 0$, are computed using the variation iteration method (VIM), the nonlinear term is expressed as infinite partial sum of Adomian polynomials substituted into the correction functional. Hence, the approximate solution is estimated for every $n \geq 0$ or at a definite $n$. In the variation iteration Adomian decomposition method (VIADM), the linear and nonlinear terms are expressed as an infinite partial sum of the components $u_n(x,t)$, $n \geq 0$, and Adomian polynomials, respectively.

The VIADM is explicit; it requires no special treatment of linearization, perturbation or discretization. The rate of convergence is far superior to the traditional variation iteration decomposition method (VIDM). We consider some numerical examples to show the rate of convergence of the VIADM as compared with VIDM and analytic solution available in the literature.

## Variation Iteration Method

In this section, we give a brief discussion of the general concept of the variation iteration method as applied to linear and nonlinear differential equations.
Let consider the general differential of the form

(1) $$Lu(x,t) + Nu(x,t) = g(x,t),$$

with prescribed auxiliary conditions, where $u(x,t)$ is unknown function, $L$ is a linear operator, $Nu$ is a nonlinear operator and $g(x,t)$ is the source term. By the variation iteration method (VIM), we construct a correction functional for (1) as follows:

(2) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)(Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s))ds, \quad n \geq 0$$

where $\lambda(s)$ is a Lagrange multiplier which can be obtained optimally via variational theory and $\delta\tilde{u}_n = 0$ is a restriction. For more on VIM, see [8-9]

## The Adomian Decomposition Method (ADM)

In this section, we consider the Adomian decomposition method explicitly.
Now, we consider the standard operator [1-4]

(3) $$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),$$

with prescribed auxiliary conditions, $u(x,t)$, an unknown function, $L$ is the highest power derivative which is easily invertible, $Nu(x,t)$ is the nonlinear term, $Ru(x,t)$ is a linear operation of order less than $L$, and $g(x,t)$ is the source term.
Applying the inverse operator $L^{-1}$ to both sides of equation (3), we obtain

(4) $$u(x,t) = L^{-1}\big(g(x,t)\big) - L^{-1}\big(Ru(x,t)\big) - L^{-1}(N(x,t)).$$

The Adomian decomposition method gives the solution as

(5) $$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \, .$$

The Adomian decomposition method uniquely defined the nonlinear term, $Nu(x,t)$ as

(6) $$Nu(x,t) = \sum_{n=0}^{\infty} A_n \, ,$$

where $A_n$ are the Adomian polynomials which are determined recursively using

(7) $$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}.$$

Let $Nu(x,t) = \alpha\big(u(x,t)\big)$, then first few Adomian polynomials are arranged as follows

$$A_0 = \alpha(u_0(x,t))$$

$$A_1 = \alpha'(u_0(x,t))\, u_1$$

$$A_2 = \alpha'(u_0(x,t))\, u_2 + \frac{u_1^2}{2!}\alpha''(u_0(x,t))$$

$$A_3 = \alpha'(u_0(x,t))\, u_3 + u_1(x,t)u_2(x,t)\alpha''(u_0(x,t)) + \frac{u_1^3}{3!}\alpha'''(u_0(x,t))$$

$\vdots$

Hence, an n-component truncated series solution is obtained as

(8) $$u_n(x,t) = \sum_{i=0}^{n} u_i(x,t)\ ,$$

where

(9) $$u_0 = L^{-1}(G) - L^{-1}(Ru),$$

(10) $$u_{n+1} = L^{-1}(G) - L^{-1}(Ru) - L^{-1}(Nu)\ ,$$

where $u_0$ is the zero component. Hence, an n-component truncated series solution is obtained as

$$u_n(x) = \sum_{i=0}^{n} u_i$$

## Variation Iteration Adomian Decomposition Method

Let the unknown function $u(x,t)$ be defined as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Then, the decomposition method [1-4] involves finding the components $u_n(x,t)\,, n \geq 0$, via the variation iteration method. Hence, the variation iteration decomposition method involves replacing the nonlinear term, $Nu(x,t)$ with $\sum_{n=0}^{\infty} A_n$, where $A_n$ are the adomian polynomials, such that we have the iterative scheme

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(Lu_n(x,s) + \sum_{n=0}^{\infty} A_n - g(x,s)\right)ds,\ n \geq 0$$

Thereafter, we replace the nonlinear term, $Nu(x,t)$ with $\sum_{n=0}^{\infty} A_n$, and the linear term with $\sum_{n=0}^{\infty} u_n$ where $u_n$ are the components obtained using the variation iteration method (VIM). Hence, we obtain the iterative scheme given as

(11) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)(L(\sum_{n=0}^{\infty} u_n) + \sum_{n=0}^{\infty} A_n - g(x,s))ds, n \geq 0$$

Equation (12) is the variation iteration Adomian decomposition scheme. Evidently, it converges better than the variation iteration decomposition method as numerical experiments reflects in the next section.

## Numerical Illustrations

In this section, we implement VIADM to solve nonlinear partial differential equations. The method is compared with VIDM and Homotopy perturbation method (HPM) [5] for efficiency and convergence. We consider problems that have analytic solutions in order to be able to obtain the error estimates and rates of convergence for each method.

**Example 5.1:**

Consider the nonlinear partial differential equation of second order

(12) $$u_{tt} - 2\frac{x^2}{t}uu_x = 0, \ t > 1,$$

with the initial conditions

(13) $$u(x,0) = 0, \quad u_t(x,0) = x.$$

The exact solution of this problem is given by

(14) $$u(x,t) = \tan(xt).$$

By the variation iteration method, we construct a correction functional for (12),

(15) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(\left(\frac{\partial^2 u_n(x,s)}{\partial t^2}\right) - 2\frac{s^2}{t}u_n(x,s)\frac{\partial u_n(x,s)}{\partial s}\right)ds, n \geq 0$$

Taking a variation on both sides of (15), we have

(16) $$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(s)\left(\left(\frac{\partial^2 u_n(x,s)}{\partial t^2}\right) - 2\frac{s^2}{t}u_n(x,s)\frac{\partial \tilde{u}_n(x,s)}{\partial s}\right)ds, n \geq 0$$

where

$$\delta N\tilde{u}_n(x,s) = u_n(x,s)\frac{\partial \tilde{u}_n(x,s)}{\partial s} = 0.$$

Hence, equation (16) can be rewritten as

(17) $$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s)\left(\left(\frac{\partial^2 u_n(x,s)}{\partial t^2}\right)\right) ds, \quad n \geq 0$$

By integration by part, we obtain

(18) $$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \left[\lambda(s)\frac{\partial u_n(x,s)}{\partial t})\right]_{s=t} - \delta[\lambda(s)u_n(x,s)]_{s=t} + \delta \int_0^t \lambda''(s)u_n(x,s)\, ds, n \geq 0$$

Finding the stationary points in (18) yields

(19) $$\lambda(s) = s - t,$$

(20) $$|\lambda(s)|_{s=t} = s,$$

(21) $$\lambda''(s) = 0$$

Equation (19) is the Lagrange multiplier and equation (20) and (21) can be identified as initial conditions.
Hence, the variation iteration method for (12) can be written as

(22) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\left(\frac{\partial^2 u_n(x,s)}{\partial t^2}\right) - 2\frac{s^2}{t}u_n(x,s)\frac{\partial \tilde{u}_n(x,s)}{\partial s}\right) ds, \; n \geq 0$$

We have the initial approximation as $u_0(x,s) = xt$. Hence for $n \geq 0$, we obtain the following approximations from using equation (22)

$$u_1(x,s) = xt + \frac{t^6}{10},$$

$$u_2(x,s) = xt - \frac{13}{10}t^6 + \frac{1}{60}t^{10},$$

$$u_3(x,s) = xt + \frac{183}{10}t^6 - \frac{19}{20}t^{10} + \frac{1}{360}t^{14}$$

$$\vdots$$

Now applying the variation iteration Adomian decomposition method, we have that

(23) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left[\frac{\partial^2}{\partial t^2}\left(\sum_{i=0}^n u_i(x,s)\right) - 2\frac{s^2}{t}\left(\sum_{i=0}^n A_i(x,s)\right)\right] ds, \; n \geq 0$$

Here, the nonlinear term is given by

$$Nu(x,t) = u(x,t)\frac{\partial u(x,t)}{\partial x}.$$

Hence, using the Adomian polynomials formulation, we obtain the following,

$$A_0 = xt^2$$

$$A_1 = xt - \frac{13}{10}t^6 + \frac{1}{60}t^{10}$$

$$A_2 = xt^3 - \frac{13}{10}t^8 + \frac{2}{75}t^{12} + x^2t^2 + \frac{1}{5}xt^7$$

$$A_3 = xt^3 - \frac{183}{10}t^8 + \frac{27}{25}t^{12} + \frac{1}{225}t^{16} + x^2t^2 - \frac{6}{5}xt^7 + \frac{1}{60}xt^{11}$$

$$\vdots$$

Solving equation (23) for $n = 2$, we obtain

(24) $$u(x,t) = xt - \frac{3}{4}t^{10} - \frac{9}{50}t^{11} + \frac{181}{10}t^6 + \frac{4}{15}t^7 + \frac{1}{225}t^{15}$$

Using (24) we compute the approximate solution $u(x,t)$ as shown in **Table 1**.

**Table 1**: Comparison of absolute error obtained by VIADM and VIDM using first approximation

| $x$ | $t = 0.015$ | | $t = 0.025$ | |
|---|---|---|---|---|
| | $E_{VIADM}$ | $E_{VIDM}$ | $E_{VIADM}$ | $E_{VIDM}$ |
| 0.10 | 9.1900e-10 | 1.1240e-09 | 7.8700e-10 | 5.1820e-09 |
| 0.30 | 3.0169e-08 | 3.0374e-08 | 1.3621e-07 | 1.4060e-07 |
| 0.50 | 1.4042e-07 | 1.4063e-07 | 6.4666e-07 | 6.5105e-07 |
| 0.70 | 3.8568e-07 | 3.8588e-07 | 1.7823e-06 | 1.7866e-06 |
| 0.90 | 8.1997e-07 | 8.2017e-07 | 3.7932e-06 | 3.7976e-06 |

**Example 5.2:**

Consider the nonlinear partial differential equation of second order

(25) $$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial (u^2)}{\partial x} - u(1-u) = 0, \;\; 0 \le x \le 1, \;\; t > 0$$

where $g(x,t) = 0$. The exact solution is

$$u(x) = e^{t-x}.$$

By the variation iteration method, we construct a correction functional for (25),

(26) $$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial u_n(x,s)}{\partial t} - \frac{1}{2}\frac{\partial\left(u_n^2(x,s)\right)}{\partial s} + u_n^2(x,s) - u_n(x,s)\right)ds, n \ge 0$$

Taking a variation on both sides of (26), we have

(27) $$\delta u_{n+1}(x,t) = \delta u_n(x,t)$$

$$+\delta\int_0^t \lambda(s)\left(\frac{\partial u_n(x,s)}{\partial t}-\frac{1}{2}\frac{\partial\left(u_n^2(x,s)\right)}{\partial s}+u_n^2(x,s)-u_n(x,s)\right)ds, n\geq 0$$

where

$$\delta N\tilde{u}(x,s)=\frac{1}{2}\frac{\partial\left(u_n^2(x,s)\right)}{\partial s}+u_n^2(x,s)=0.$$

Hence, equation (27) can be rewritten as

(28) $$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(s)\left(\frac{\partial u_n(x,s)}{\partial t}-u_n(x,s)\right)ds,\ \ n\geq 0$$

By integration by part, we obtain

(29) $$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta[\lambda(s)\partial u_n(x,s)]_{s=t}-\delta\int_0^t \lambda'(s)u_n(x,s)\,ds, n\geq 0$$

Finding the stationary points in (29) yields

(30) $$\lambda(s)=-1$$

(31) $$\lambda'(s)|_{s=t}=0$$

Equation (30) is the Lagrange multiplier and equation (31) can be identified as initial conditions.
The correction functional for equation (25) becomes

(32) $$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,s)}{\partial t}-\frac{1}{2}\frac{\partial\left(u_n^2(x,s)\right)}{\partial s}+u_n^2(x,s)-u_n(x,s)\right)ds, n\geq 0$$

We take initial approximation as

$$u_0(x,t)=e^{-x}.$$

By the variation iteration Adomian decomposition method, equation (32) can be written as

$$u_{n+1}(x,t)=u_n(x,t)$$
$$-\int_0^t\left[\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}u_n\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}A_n\right)-\sum_{n=0}^{\infty}u_n+\sum_{n=0}^{\infty}A_n\right]ds,\ \ n\geq 0$$

Here, the nonlinear term is given by

$$Nu(x,t)=\frac{1}{2}\frac{\partial(u^2)}{\partial x}+u^2.$$

Hence, using the Adomian polynomials formulation above, we obtain the following,

$$A_0=\frac{1}{2}\frac{\partial(u_0^2)}{\partial x}+u_0^2,$$

$$A_1 = \frac{d}{dx}\left(\frac{1}{2}\frac{\partial(u_0^2)}{\partial x} + u_0^2\right)u_1,$$

$$A_2 = \frac{d}{dx}\left(\frac{1}{2}\frac{\partial(u_0^2)}{\partial x} + u_0^2\right)u_2 + u_1^2,$$

$$\vdots$$

Using the above relations for $n \geq 0$, we obtain

$$u_1(x,t) = e^{-x} + e^{-x}t,$$

$$u_2(x,t) = e^{-x} + e^{-x}t + \frac{1}{2}e^{-x}t^2,$$

$$u_2(x,t) = e^{-x} + e^{-x}t + \frac{1}{2}e^{-x}t^2 + \frac{1}{6}e^{-x}t^3,$$

$$\vdots$$

$$u(x,t) = e^{-x} + e^{-x}t + \frac{1}{2}e^{-x}t^2 + \frac{1}{6}e^{-x}t^3 + \frac{1}{24}e^{-x}t^4 + \cdots = e^{(t-x)}.$$

which is same result obtained using HPM [5].

## Conclusion

The variation iteration Adomian decomposition method has been successively implemented for resolving nonlinear partial differential equations. In **example 5.1**, the mode of convergence varies with the parameter t as clearly shown in Table 1. While, in **example 5.2**, the solution converges to the exact solution for $n \geq 0$, which coincides with the same result obtained in [5] using HPM. The method is very effective with no requirement for linearization or discretization. Hence, the method is favourably recommendation to other areas in applied mathematics. All the computations were carried out with the aid of Maple 18 software.

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