Comparing Approximations of a Wavelet Collocation Method of Various Wavelets

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Abstract

In this paper we describe the application of a wavelet collocation method on different ODE's. Here we compare the approximation error of various Wavelets. The Shannon wavelet and the Meyer wavelet provides very good results. This method can be extended to unstable and stiff differential equations. In this work we also show how to set the Parameters of the collocation method and we present a general algorithm for this method.

Introduction

As part of a research project we investigated how to determine the optimal parameters for a wavelet collocation method. In the classical approach to collocation methods the approximation function is based on polynomials. These methods are equivalent to implicit Runge-Kutta method, which are used in stiff problems and boundary value problems. In the wavelet collocation method the approximation functions are constructed by a wavelet base.

There are a lot of different parameters to be set which brings up the question how useful the approximation function is if the exact solution is unknown. Here, we performed a series of simulations in which a criterion was found which theoretical could be used for a estimation. Using regression analysis, there were significant correlations between this criterion and the mean square approximation error. The criterion for evaluating the approximation of the herein described wavelet collocation method was used and theoretically justified by an estimate in [16] by M. Schuchmann.

In this study, various wavelets were compared, because there are whole families of wavelets, such as Daubechies wavelets, the Meyer wavelets or the Battle Lemarié wavelets available. One wavelet that does not have compact support, and not even a high order, provided very good approximations. The approximation functions can even be used to for extrapolations. This wavelet was the Shannon wavelet, which is infinitely differentiable and, unlike many other wavelets has a mother wavelet and a scaling function (also called father wavelet) which can be written in closed form.

We will use an approach in which the trial function is composed of a wavelet basis. Instead of solving a system of equations and to set the residuals equal to zero at certain points, we minimize the sum of squared residuals (at the collocation points), so that we are not restricted in the number of collocation points.

The advantage of the wavelet collocation method is that like other collocation method it also can be applied to stiff differential equations. Moreover, it can even be used in non-stable problems (see [15]). As an approximation we not only get points but an approximation function. Compared to other collocation method, for example, based on polynomials (see [3]), one can also cover a larger interval with an approximation, i.e. you do not have to use composite functions for small subintervals. These are the advantages of a wavelet collocation method as well as using the approximation for extrapolation outside the original approximation interval. As a disadvantage, you could argue that if there is a differential equation for which one needs no boundary value problem methods (i.e. if they are not stiff or unstable) more computing time may be needed since a minimization problem or a system of equations must be solved.

There are wavelets which are called interpolating wavelets with special properties. There are a number of publications on these wavelets. These deal with error estimates and also with the approximation of the solutions of initial value problems and boundary value problems (for ordinary and partial differential equations), see [23] and [4], as well as with the sinc collocation (see [5], [1], [10]) with special support points ("sinc grid points", see [10]). The scaling function of the Shannon wavelets which we use later are based on the sinc function and also have interpolating properties (see [18]).

In [9] a quasi-Shannon wavelet is used to approximately solve a boundary value problem (with second-order ordinary differential equation). The scaling function of the Shannon wavelets is weighted by a Gaussian function, so that the decay of the scaling function is improved.

Error estimates are provided for the Shannon wavelet (i.e. a sinc collocation with a transformation) in [10] and [1]. For interpolating wavelets estimates can be found in [19] and [6].

The method described here can also be applied to partial differential equations (see [17]). In addition, the method can also be used for parameter identification. For that an estimate in two steps was tested by us and M. Schuchmann has developed an error estimate, which was published by us in [18].

In the wavelet theory a scaling function ϕ is used, which belongs to a MSA (multi scale analysis). From the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a the sequence of subspaces with the following property:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k\in Z}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

We use the following approximation function

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t) \quad \text{, with } \phi\in C^r(R).$$

k_{max} and k_{min} depend on the approximation interval $[t_0, t_{end}]$ (see [7]). r is the order of the ODE.

Now we can approximate the solution of an initial value problem $y' = f(y,t)$ and $y(t_0) = y_0$ by minimization of the following function

$$(1) \qquad Q(c) = \sum_{i=1}^{m} \left\| y_j'(t_i) - f(y_j(t_i), t_i) \right\|_2^2 + \left\| y_j(t_0) - y_0 \right\|_2^2 \ .$$

For $m = |k_{max} - k_{min}|$ we get an equivalent problem:

$$y_j'(t_i) = f(y_j(t_i), t_i) \text{ for } i = 1, 2, ..., m \text{ and } y_j(t_0) = y_0 .$$

We will use equidistant points or collocation points t_i with $t_i = t_0 + i \cdot h$ and

$$h = \frac{t_{end} - t_0}{m} .$$

To detect large residuals in other places as the collocation points, we have a further value used for comparison with Q_{min} (here in y_j the vector c will be set to the value in the minimum of Q, see (1)).

$$Q_a = \sum_{i=1}^{m_a} \left\| y_j'(\tau_i) - f(y_j(\tau_i), \tau_i) \right\|_2^2 + \left\| y_j(t_0) - y_0 \right\|_2^2$$

with $\tau_i = t_0 + i \cdot h/a$. $m_a = a \cdot m$ with $a > 1$ as an integer. Since the wavelet collocation method provides a whole approximation function y_j and not only points, we can calculate Q_a without additional effort. If $Q_a >> Q_{min}$ (and Q_{min} was very small) then m (the number of collocation points) should be increased. When comparing Q_{min} with Q_a, Q_a should be weighted by $1/a$ if a is large. In the simulations $a = 2$ proved sufficient.

Q_a can additionally be justified by an error estimation of the residuals at theoretically any number of points. This was derived by M. Schuchmann. In this error estimate a certain value occurs as a factor. Q_a represents the Riemann sum for this value i.e. this can be approximated by Q_a. For this we use the following theorem:

Theorem:
Let $y' = f(y, t)$ with $y(t_0) = y_0$ and let (for $t \geq t_0$)

$$||y_j'(t) - f(y_j(t), t)|| \leq M(t),$$

$$||f(y(t), t) - f(y_j(t), t)|| \leq l(t) \cdot ||y(t) - y_j(t)|| \text{ with } l(t) > 0$$

and

$$y_j(t_0) = y_0.$$

With

$$L(t) = \int_{t_0}^{t} l(s)ds$$

follows (for $t_{end} \geq t_0$):

$$\left\| y_j(t_{end}) - y(t_{end}) \right\| \leq e^{L(t_{end})} \cdot \left\| e^{-L} \right\|_{L^2([t_0, t_{end}])} \cdot \left\| M \right\|_{L^2([t_0, t_{end}])}$$

The proof can be found in [16]. The factor on the right hand side of the inequality can now be approximated with Q_a by

$$\|M\|_{L^2([t_0,t_{end}])} \approx \sqrt{\frac{t_{end}-t_0}{m_a}Q_a}$$

Analogous we could treat boundary conditions instead of the initial condition. This method can be even used analogous for PDEs, ODEs of higher order or ODEs, which have the Form $F(y',y,t)=0$.

If we have a second Order ODE

$$F(y'',y',y,t)=0$$

with boundary conditions

$$y(t_0) = y_0$$
and
$$y(t_{end}) = y_{end}$$

like in the following example, we minimize

$$Q(c)=\sum_{i=1}^{m}\left\|F(y_j''(t_i),y_j'(t_i),y_j(t_i),t_i)\right\|_2^2+\left\|y_j(t_0)-y_0\right\|_2^2+\left\|y_j(t_{end})-y_{end}\right\|_2^2 \quad .$$

Analogous we treat conditions of the form

$$y(t_0) = y_0$$
and
$$y'(t_0) = y'_0$$

Comparing the Orthogonal Projection of y in V_j

Now we want to approximate two functions in the following two examples, which are not quadratic integrable on R.

Example I:
We begin with an approximation of the function $y(t) = e^{-t}$ on $I = [0, 2]$. y is not in $L^2(R)$, but every on I continuous function is in $L^2(I)$ or $1_I y$ (with indicator function 1_I of I) is in $L^2(R)$. So we set $k_{max} = -k_{min} = 20$ and we calculate an approximation function by an orthogonal projection from $1_I y$ on V_3. Therefore we calculate the coefficients of the approximation function over a scalar product (compare [v5]):

$$c_k = \left\langle 1_{[0,2]}\, y, \phi_{j,k} \right\rangle = \int_0^2 y(t)\cdot\phi_{1,k}(t)\,dt$$

With the Shannon wavelet we get a worse approximation (dashed line is thee graph of *y*). We consider in the graph only the interval [0.5,1] because of we cut the original function y at the edges we get worse approximation at the edges.

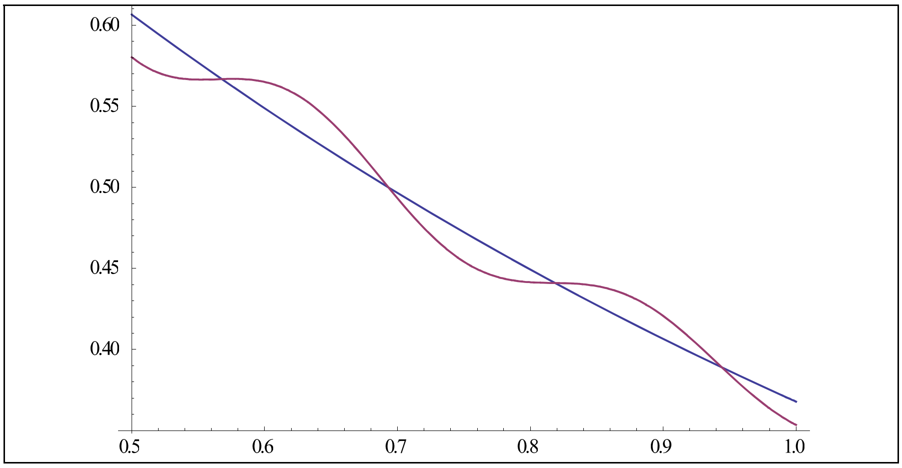

Figure 1. Graphs of y_3 (orthogonal projection form $I_I y$ on V_3) and y

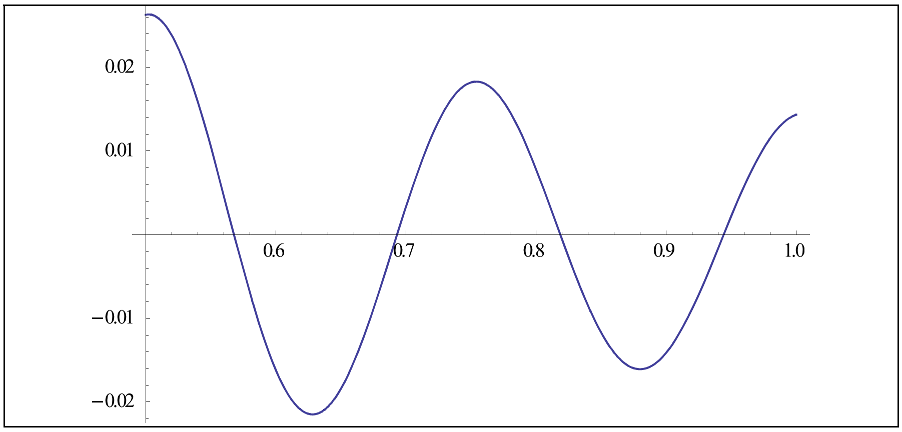

Figure 2. Graphs of $y - y_3$

With the Daubechies wavelet of order 8 we get no good approximation, but better approximation on $I = [0; 2]$, but in the midle of the interval I:

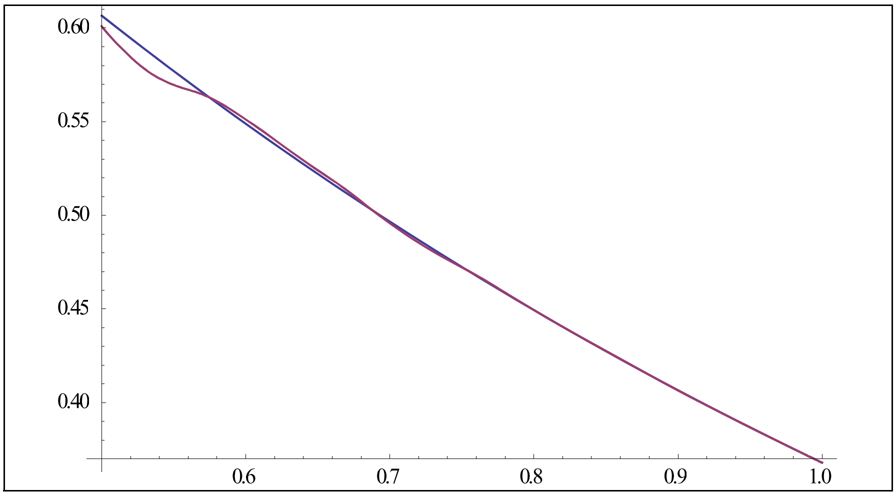

Figure 3. Graphs of y_3 (orthogonal projection form $I_I y$ on V_3) and y, Daubechies wavelet order 8

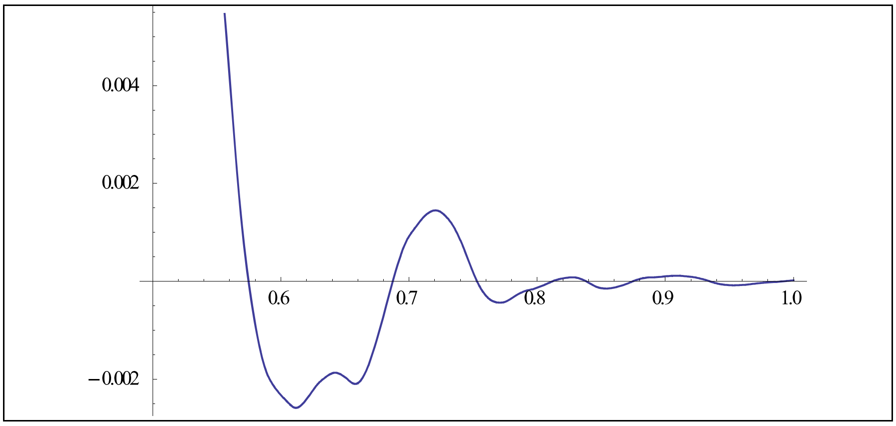

Figure 4. Graphs of y - y_3, Daubechies wavelet order 8

But the approximation with the wavelet collocation method can be much better with the Shannon wavelet, what we see in the following examples in many examples too.

Now we calculate the coefficients c_k by the minimization of Q (see (1)). We use the initial value problem $y' = -y$, $y(0) = 1$ and set even $j = 1$. We use the collocation points $t_i = i/20$ with $i = 0, 2, ..., 40$ and the Shannon wavelet.

Graphically we see no difference between the approximation function y_1 and y on I:

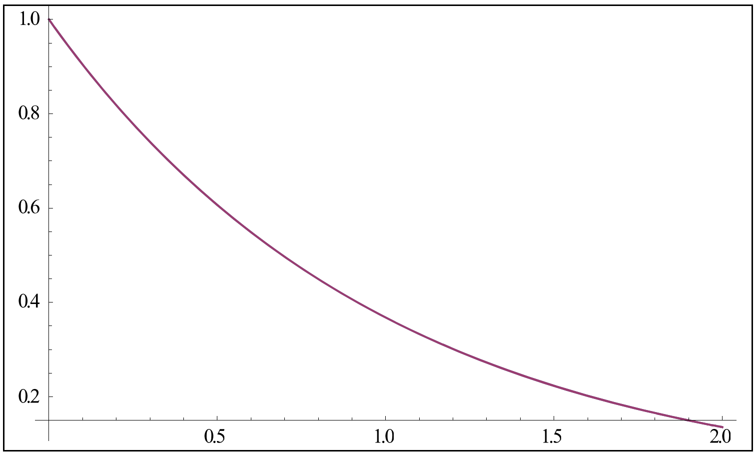

Figure 5. Graphs of y_1 (calculated by min Q) and y

Here is the graph of the difference function $y_j - y$:

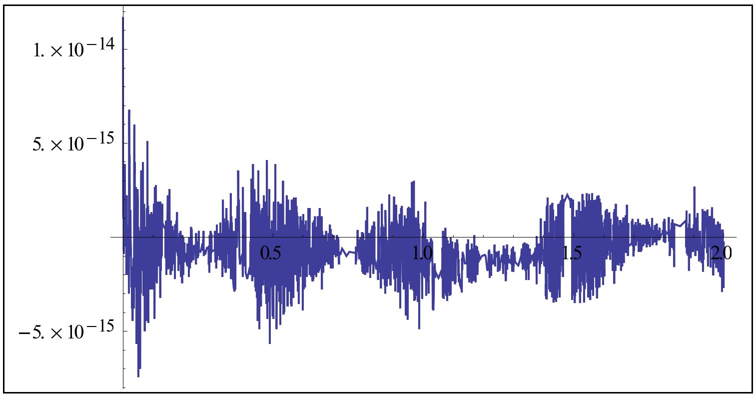

Figure 6. Graph of y_1 - y (y_1 calculated by min Q) and y

With the Daubechies wavelet of order 8 (D8) we get the following graph:

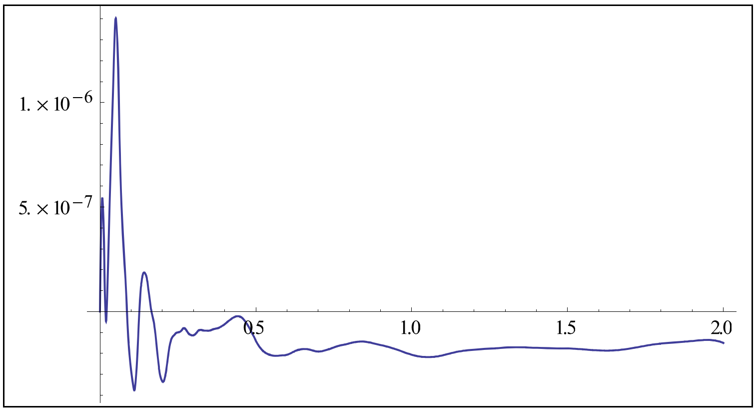

Figure 7. Graph of y_1 - y (y_1 calculated by min Q) and y, with D8

Example II: Boundary value problem with a second order differential equation

Consider the following boundary value problem:

$$F(y'', y', y, t) = y'' - 1/\zeta \cdot (y - (\zeta \cdot \pi^2 + 1)\cos(\pi \cdot t)) = 0 \text{ with } y(-1) = 0 \text{ and } y(1) = 0,\ \zeta > 0.$$

This example was also used in the chapter with the title "comparing different wavelets" (Sample 8) and was found as test problem 14 on the website of Jeff Cash (Imperial College, London). If we write the problem as a first order system, with $y_1 = y'$ and $y_2 = y$, then

$$y_1' = 1/\zeta \cdot (y_2 - (\zeta \cdot \pi^2 + 1)\cos(\pi \cdot t))$$
$$y_2' = y_1$$

respectively
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1/\zeta \\ 1 & 0 \end{pmatrix}}_{=A} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -1/\zeta \cdot (\zeta \cdot \pi^2 + 1)\cos(\pi \cdot t) \\ 0 \end{pmatrix}.$$

The matrix A has the eigenvalues $\lambda_{1/2} = \pm \frac{1}{\sqrt{\zeta}}$.

Thus we see that at small ζ the solution function is composed of a function with a steep incline and a sharply decreasing function, which can lead to problems with numerical methods.

We are looking for an approximation on the interval $I = [-1, 1]$ and set $\zeta = 0.01$. We minimize

$$Q(c) = \sum_i (F(y_j''(t_i), y_j'(t_i), y_j(t_i), t_i))^2 + (y_j(-1) - 0)^2 + (y_j(1) - 0)^2$$

with the collocation points $t_i = i \cdot h$ (mit $i = 1, 2, \ldots, m$, $m = r \cdot k_{max}$), with $h = 2/(r \cdot k_{max})$ and $k_{min} = -k_{max}$. We use again $k_{max} = 15, 20, 25$, $r = 1, 2, 3$ and $j = 0, 1, 2$.

Q_a is defined here in analogy to (5) for this boundary value problem:

$$Q_a = \sum_i (F(y_j''(\tau_i), y_j'(\tau_i), y_j(\tau_i), \tau_i))^2 + (y_j(-1) - 0)^2 + (y_j(1) - 0)^2$$

In the graphs the mean square error *mse* is again displayed with

$$mse = \frac{1}{101} \sum_{i=0}^{100} (y(2i/100 - 1) - y_j(2i/100 - 1))^2 \quad .$$

We start with the Shannon wavelet. Below are the graphs of the approximations that were relatively goodwith following triples (j, k_{max}, m): (0, 20, 40), (1, 15, 30), (1, 20, 40) and (2, 25, 50). It is noticable that here $r = 2$ i.e. $m = 2 \cdot k_{max} = |k_{max} - k_{min}|$. Below are the graphs of y_j , y and $y_j - y$.

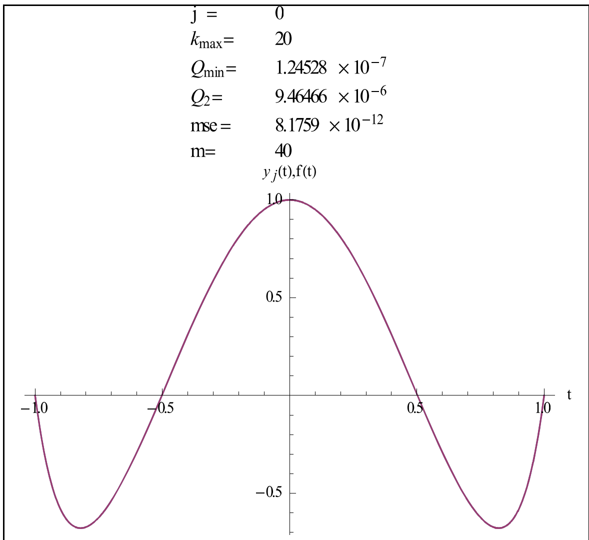

Figure 8

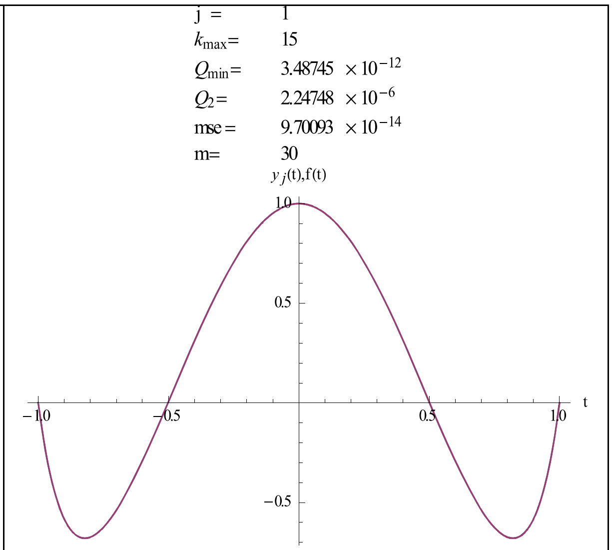

Figure 9

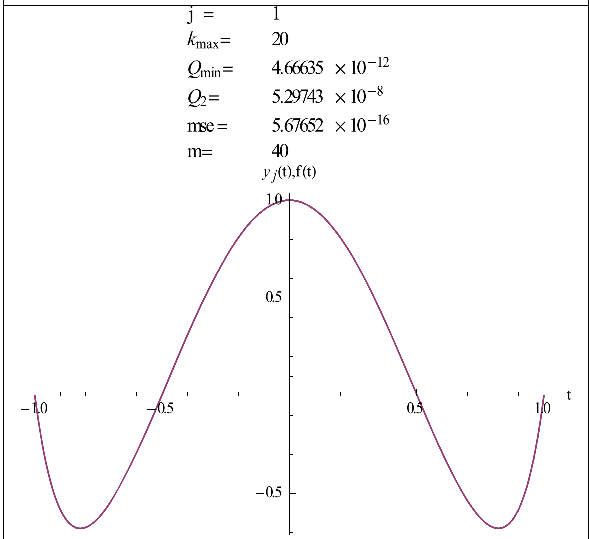

Figure 8

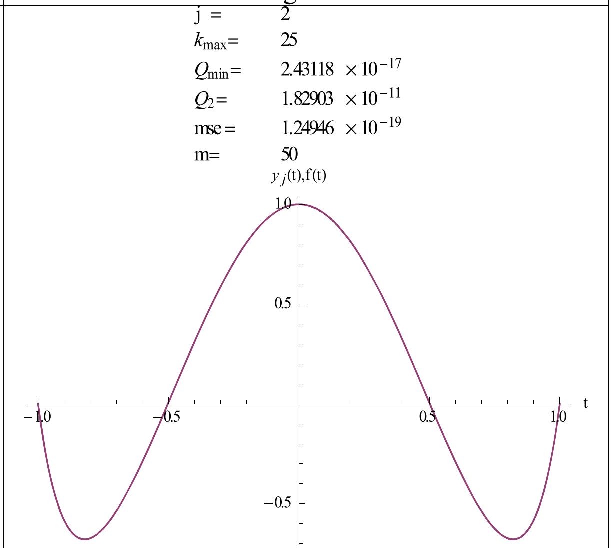

Figure 9

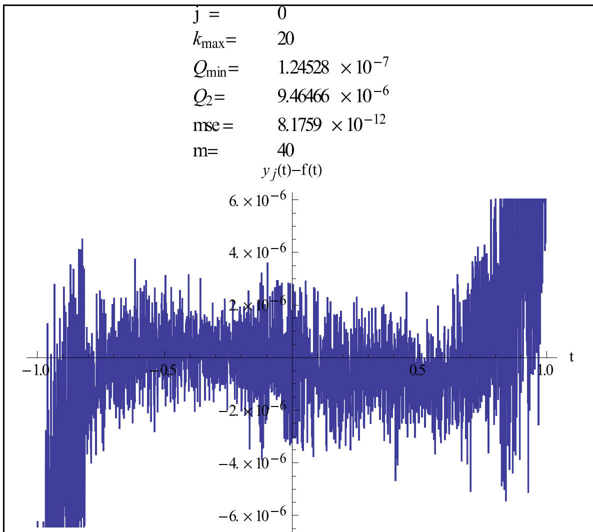

Figure 10

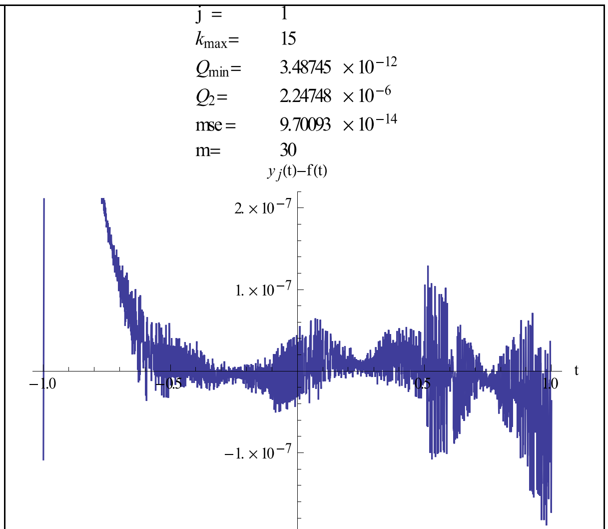

Figure 11

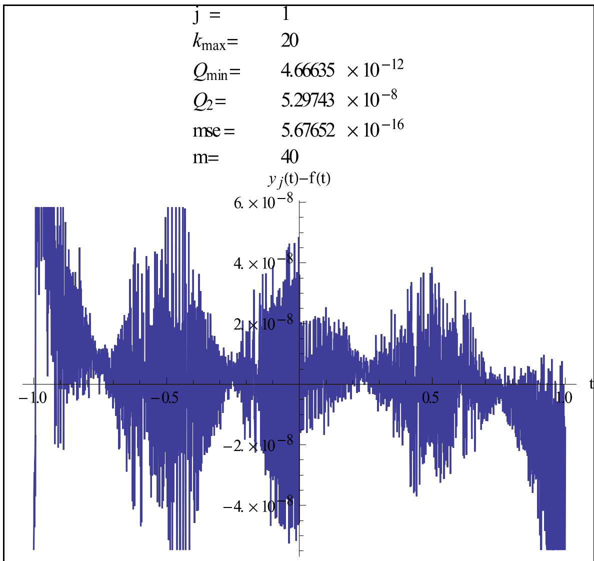

Figure 12

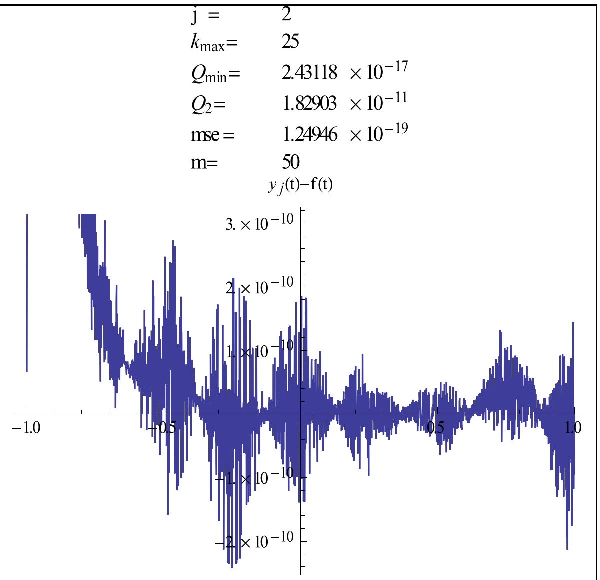

Figure 13

With the points *(-ln(Q_{min}), -ln(mse))* (with various *j*, k_{max} and *r*) a regression is calculated. Here again we see a relationship. However, more clearly the relationship is seen in the regression with the points *(-ln(Q_2), -ln(mse))*.

Regression with the points *(-ln(Q_{min}), -ln(mse)):*

	Estimate	SE	TStat	PValue
1	6.14361	1.34266	4.57571	0.00011192
x	1.03175	0.0595142	17.3362	1.9112×10^{-15}

, RSquared → 0.923205

Table 1

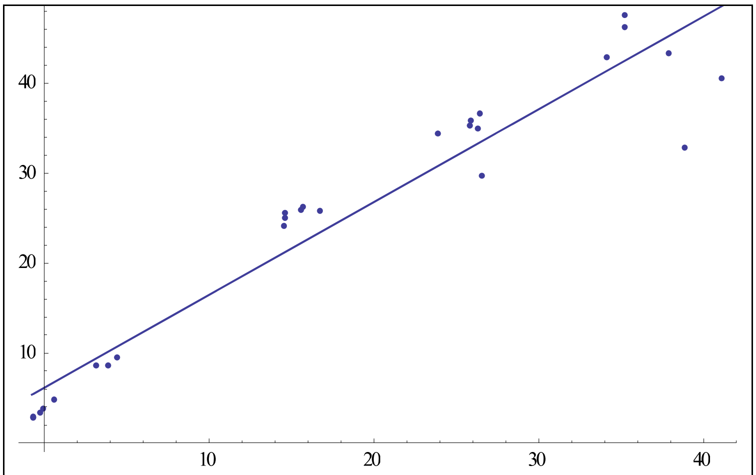

Figure 14

Regression with *(-ln(Q_2), -ln(mse)):*

	Estimate	SE	TStat	PValue
1	8.12563	0.801252	10.1412	2.41445×10^{-10}
x	1.41116	0.0514383	27.434	3.50593×10^{-20}

, RSquared → 0.967851

Table 2

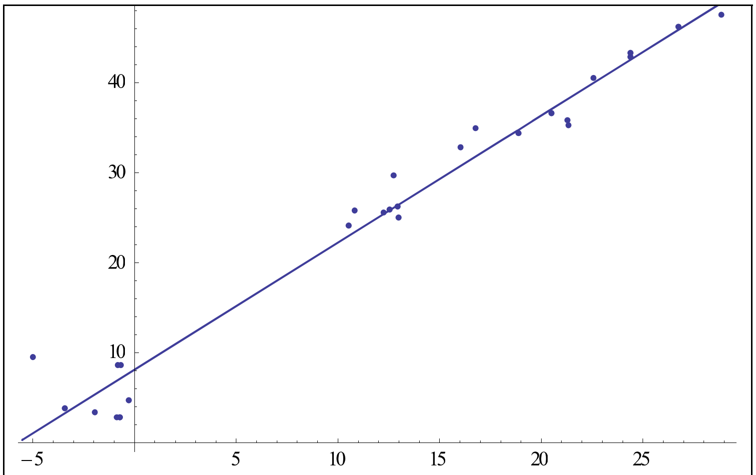

Figure 15

Now for comparison follows a regression of *-ln(Q_2)* to *-ln(mse)* using the Meyer wavelet:

	Estimate	SE	TStat	PValue
1	8.54055	0.747398	11.427	2.03499×10^{-11}
x	1.45183	0.0503363	28.8426	1.04043×10^{-20}

, RSquared → 0.970825

Table 3

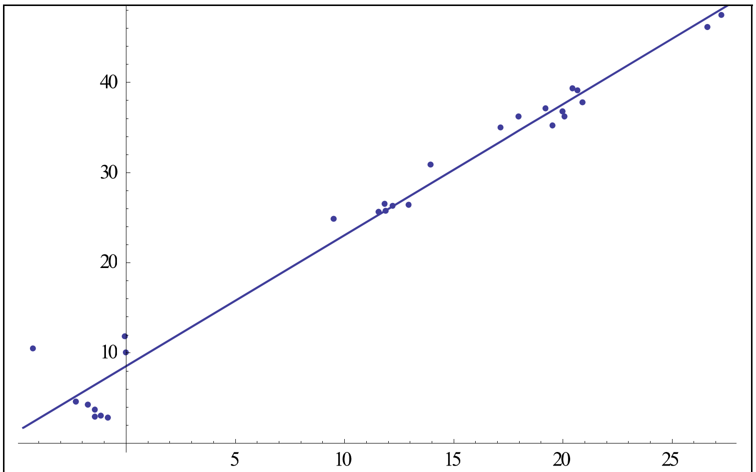

Figure 16

The Daubechies wavelet of order 8 as well as the Battle-Lemarié wavelet provided no useful approximations. This could be recognized by a very high Q_{min} and Q_2 (see the chapter with the title "Comparison of different wavelets").

Even the NDSolve function of Mathematica 8 (also Mathematica 9) has problems with this boundary value problem.

There is a note displayed:
NDSolve::berr: There are significant errors _{4.85642×10^{-29},-1.83451×10^{-6}}_ in the boundary value residuals. Returning the best solution found.

Here is the graph of the solution curve computed by NDSolve. At t = 1 we see major deviations:

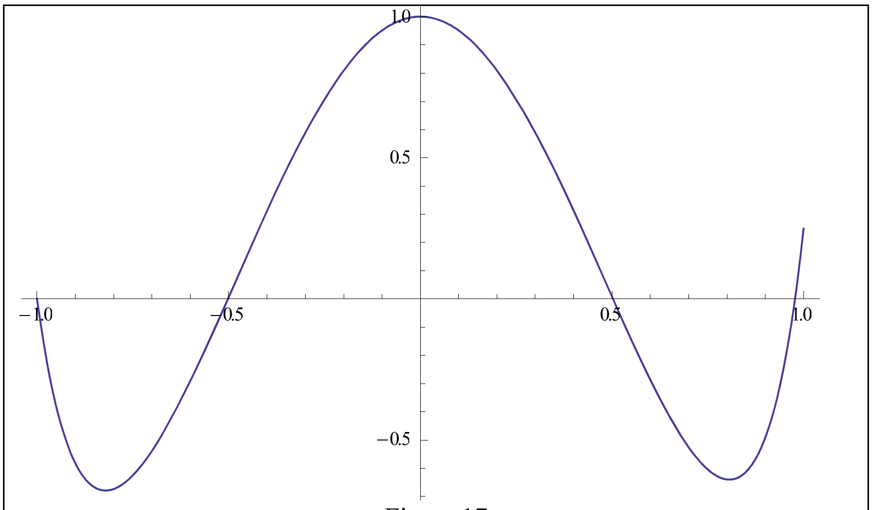

Figure 17

If for example $\zeta = 0.001$ is set, then the problems get even more severe:

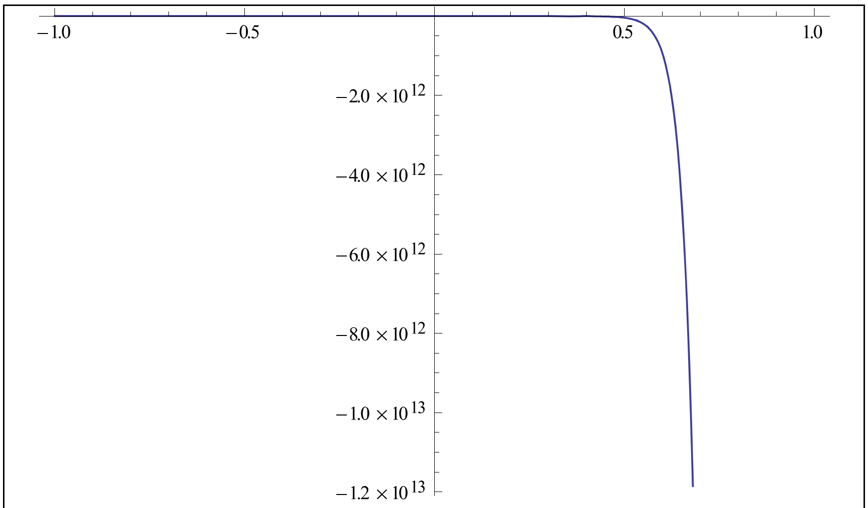

Figure 18

Mathematica displayed following note here:
NDSolve::bvluc: The equations derived from the boundary conditions are numerically ill-conditioned. The boundary conditions may not be sufficient to uniquely define a solution. The computed solution may match the boundary conditions poorly.
NDSolve::berr: There are significant errors _{-1.10934×10^{-30},-4.30118×10^{11}}_ *in the boundary value residuals. Returning the best solution found.*

With this smaller ζ the wavelet collocation method has no problems, but it can cause big deviations between y_j and y in the neighbourhood of $t = -1$ and $t = 1$ (i.e. in the vicinity of the interval limits of the approximation interval I) provided j is too small. This is due to the relatively large slope of y in this area. For this reason the collocation points $t_i = i{\cdot}h$ should begin with $i = 0$, so that the slope at $t = -1$ is considered in Q.

With a smaller j also relatively large values of Q_2 can occur; even if the whole approximation (or without the areas at the edge of approximation interval I) is good. This is due to the fact that $d(t) = (F(y_j''(t), y_j'(t), y_j(t), t))^2$ becomes relatively large in the aforementioned neighbourhood.

With such types of functions the points τ_i on the edge of the approximation area could be left out, if a good approximation on the inner part of the interval I is needed and this approximation should be identified with Q_2.

Now we set $\zeta = 0.001$ and we minimize Q. Below are the graphs of y_j and y and the graph of y_j - y for $j = 3$ and $r = 3$:

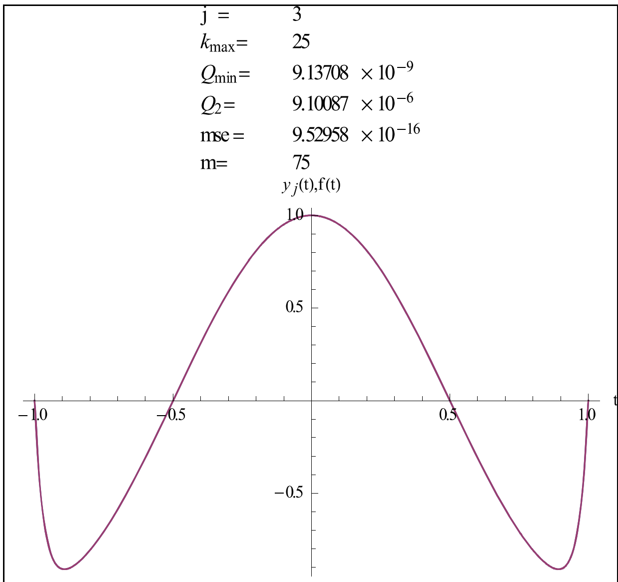

Figure 19

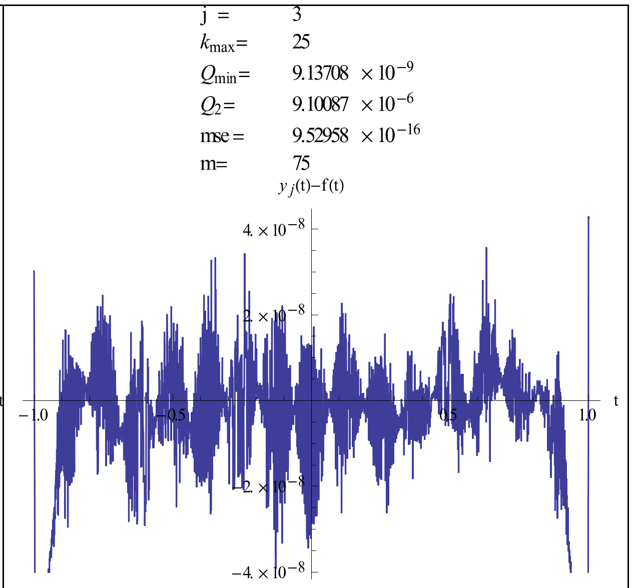

Figure 20

If j is too small there are problems on the edges and Q_{min} still relatively large. For example when $j = 1$ and $k_{max} = 15$:

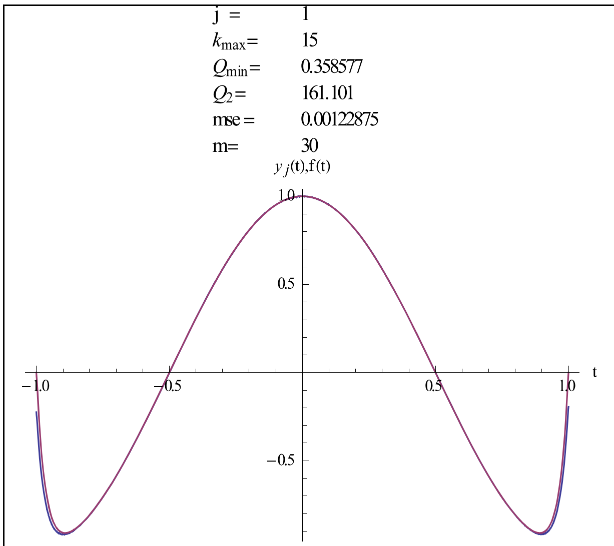

Figure 21

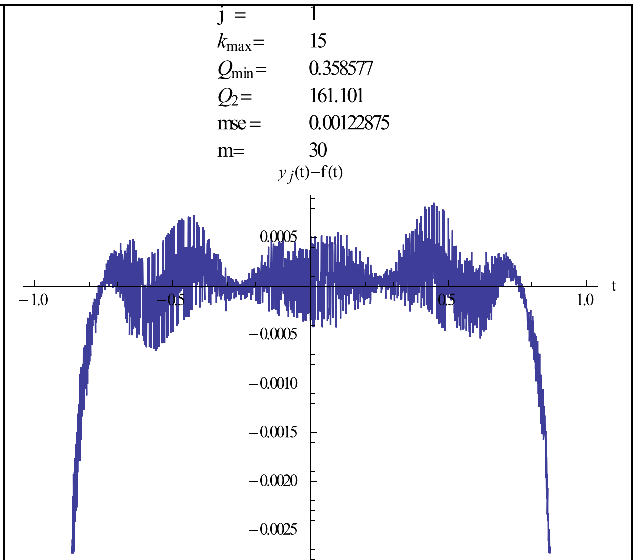

Figure 22

Or when $j = 1$ and $k_{max} = 25$:

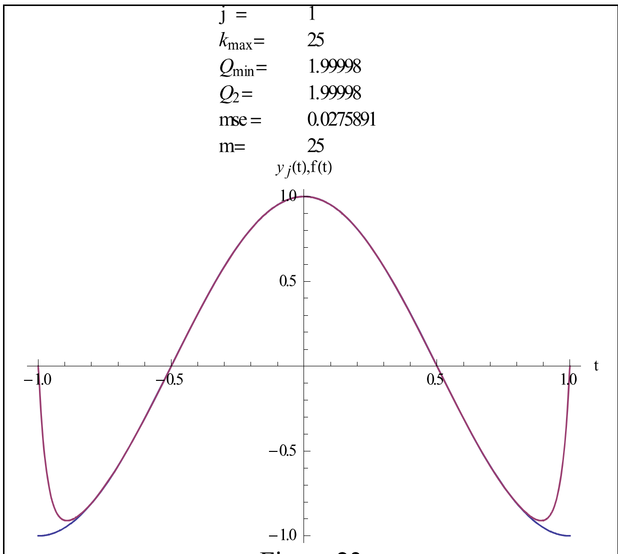

Figure 23

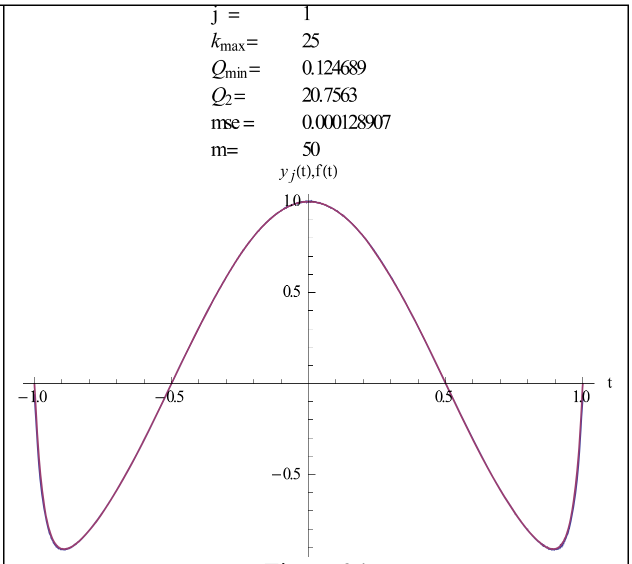

Figure 24

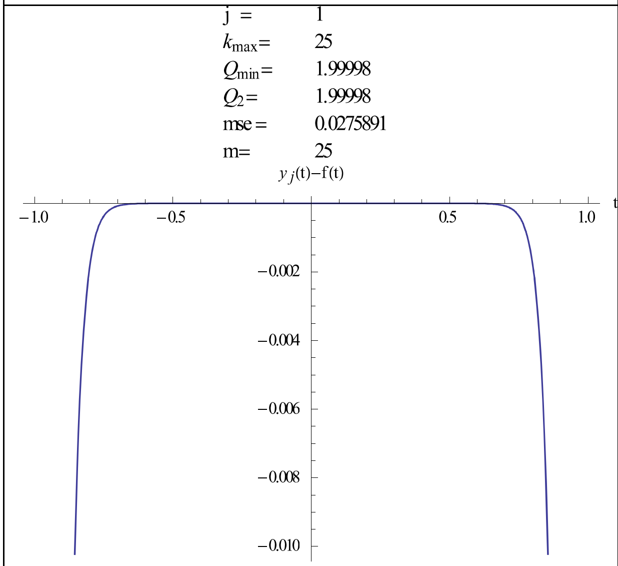

Figure 25

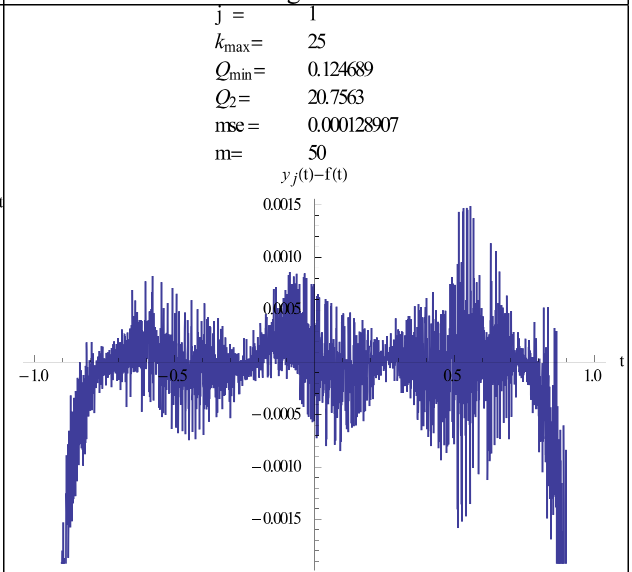

Figure 26

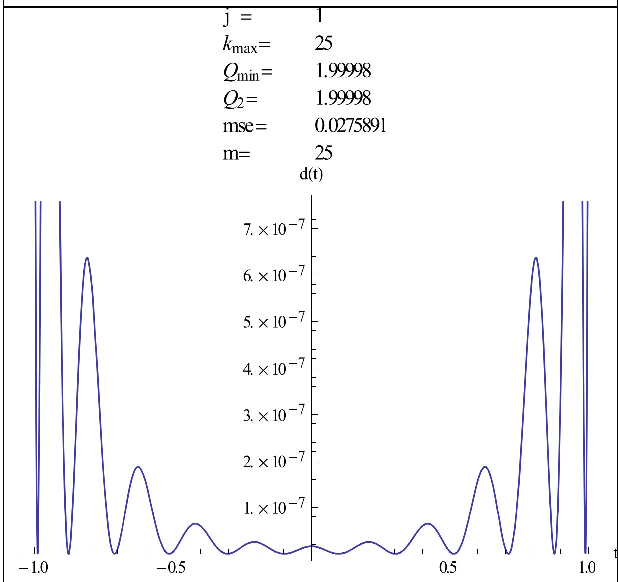

Figure 27

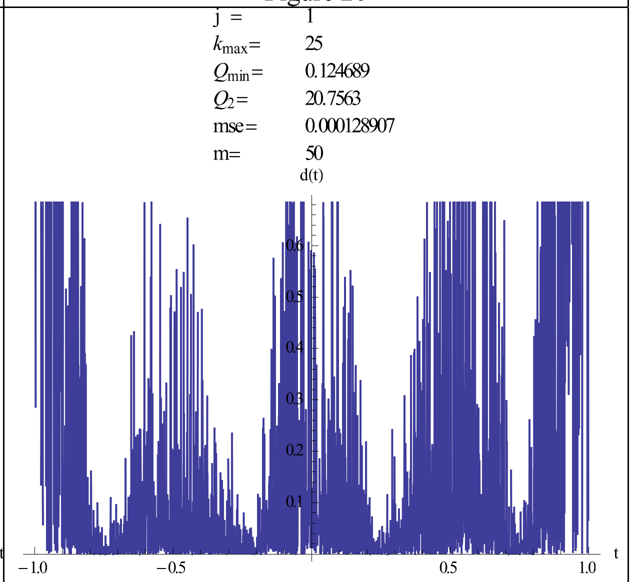

Figure 28

The Algorithm

If no further information is available, one can start with j = 1 and $m = |k_{max} - k_{min}|$ and minimize Q. k_{min} and k_{max} should be tuned to the approximation interval. Suitable positive real numbers ϵ_1 and ϵ_2 should be chosen.

If $Q_{min} < \epsilon_1$, then it is checked wether $Q_a < \epsilon_2$ applies (with $a > 1$, for example a = 2). If both conditions are met, then the iteration is finished.

If $Q_{min} < \epsilon_1$ is not met, j is incremented by 1 (if a sufficient number of basis functions $\phi_{j,k}$ are chosen with respect to the approximation interval I).

If $Q_{min} < \epsilon_1$ is met but $Q_2 < \epsilon_2$ not, then m should be increased.

Remark:
1) For the Shannon wavelet j = 1 was sufficient for most simulations. If steep slopes or large curvatures are present, good approximations where were calculated with j = 2 or j = 3. Here you can also start with a larger m.

2) Since k_{max} and k_{min} also depend on j (i.e. for bigger j a bigger k_{max} and smaller k_{min} is needed), with a bigger j automatically a bigger m should be chosen. You could double the value of m when j rises by 1. This rule could be useful in relation to the Shannon wavelet, taking into account the sampling frequency of the Shannon theorem.

3) Minimizing Q instead of solving the equation system (2) has several advantages. One can use more collocation points and the least squares method is used to calculate the parameters c_k, because the differential equation is generally (if y_j is not the exact solution) only approximately fulfilled (but the residuals are very small with good approximations). Moreover, the equations (2) several examples have been in ill-conditioned.

4) If y has near the beginning big slopes or curvatures as with some stiff differential equations and only a good approximation in the interior of the interval I is needed, then only $\tau_i \in [\tilde{t}_0, t_{end}]$ with $\tilde{t}_0 > t_0$ ($[\tilde{t}_0, t_{end}]$ is part of the overall approximation interval $I = [t_0, t_{end}]$) is sufficient for the calculation of Q_a. In this case the summation index in (5) does not start with $i = 1$ (e.g. $i = a$).

5) Although the Shannon wavelet does not have compact support, and no high order, but it returned in the simulations often significantly better results than other wavelet (even at relatively small $|k_{max} - k_{min}|$). In addition, it has several advantages for use in an approximation:

(a) The scaling function (as well as the wavelet) is defined analytically.

(b) The scaling function is many times continuously differentiable (see [11]).

(c) The scaling function is band limited and you can use this with the sampling theorem of Shannon, and thus "generalize" (see Remarks 1). This gives you information about the choice of j in Fourier space.

Comparing different wavelets

Finally we compare the approximation behaviour of different wavelets (Shannon, Daubechies of order 8, Meyer of order 3 and Battle-Lemarié of order 5). We minimize Q and use the collocation points $t_i = i \cdot h$ (with $i = 1, 2, \ldots, m$; $m = r \cdot k_{max}$), with $h = 2/(r \cdot k_{max})$ and $k_{min} = -k_{max}$. It was $k_{max} = 15, 20, 25$, $r = 1, 2$ and $j = 0, 1, 2$ used.

Example 1: $y' = -t\,y$, $y(0) = 1$, $I = [-1, 1]$
Example 2: $y' = -2ty^2$, $y(0) = 1$, $I = [-2, 2]$
Example 3: $y' = -y - 2y^3 + \sin(2t)$, $y(0) = 0$, $I = [0, 4]$
Example 4: $y' = y - 2t/y$, $y(0) = 1$, $I = [0, 4]$

Example 5: $y'' = -y' - 257/4y$, $y(0) = 0$ and $y'(0) = 8$, $I = [0, 4]$
Example 6: $y'' = -100\,y$, $y(0) = 0$ and $y'(0) = 10$, $I = [0, 4]$
Example 7: $y'' = 3/2y^2$, $y(0) = 4$ and $y(1) = 1$, $I = [0, 1]$
Example 8: $y'' = 1/\zeta \cdot (y - (\zeta \cdot \pi^2 + 1)\cos(\pi \cdot t))$, $y(-1) = y(1) = 0$, $\zeta = 0.01$, $I = [-1, 1]$.
Example 9: $y'' = -ty'/\gamma$, $y(-1) = 0$ and $y(1) = 2$, $I = [-1, 1]$, $\gamma = 0.1$

We now compare the mean values $ln(Q_{min})$, $ln(Q_2)$ and $ln(mse)$. The mean values were formed over the logarithmic values.

Wavelet	mean values $\ln(Q_{min})$	$\ln(Q_2)$	$\ln(mse)$
Shannon	-23.0799222	-12.9773489	-20.0139956
Daubechies	-11.1367372	3.67751667	-7.689269
Meyer	-26.1509444	-14.1988722	-21.4952933
Battle-Lemarié	-10.9775212	4.46392222	-4.58829762

Table 4

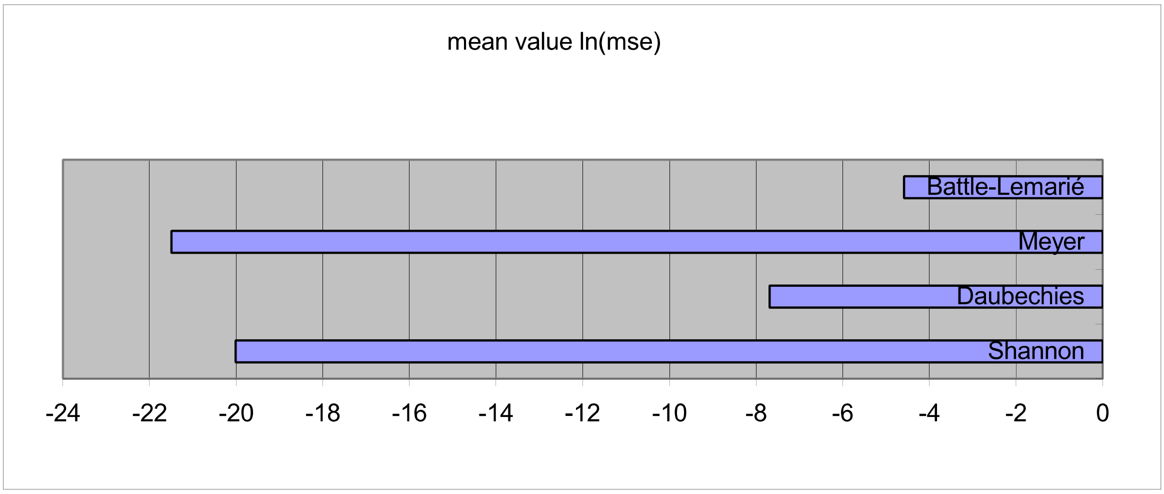

Figure 29

		Mean value			median			std. deviation		
ex.	Wavelet	$\ln(Q_{min})$	$\ln(Q_2)$	ln(mse)	$\ln(Q_{min})$	$\ln(Q_2)$	ln(mse)	$\ln(Q_{min})$	$\ln(Q_2)$	ln(mse)
1	Shannon	-45.1416	-34.9871	-43.1020	-59.5881	-32.7114	-40.7852	20.5402	18.4473	18.4535
	Daubechies	-22.6407	-8.5528	-19.5461	-18.2668	-10.3142	-19.8111	15.8459	5.6637	3.5565
	Meyer	-44.5049	-36.9365	-44.6782	-45.2432	-33.3143	-42.1668	24.3334	20.8302	20.6762
	Battle-Lemarié	-24.7375	-12.4429	-19.7694	-17.9303	-12.3960	-20.4020	16.1280	1.8081	1.9071
2	Shannon	-23.3086	-9.4866	-17.4577	-21.3585	-10.0678	-18.2280	12.5373	6.2265	6.7762
	Daubechies	-7.6038	4.9098	-5.9686	-5.5510	3.2476	-6.3710	6.9106	9.8605	4.7916
	Meyer	-28.3080	-10.2096	-18.3243	-25.9299	-8.7265	-16.5092	15.2511	7.5339	8.2264
	Battle-Lemarié	-21.9864	-4.1248	-11.4590	-17.3911	-3.5546	-12.0687	16.0386	4.0742	4.3339
3	Shannon	-17.4368	-7.4721	-13.8434	-16.9744	-7.9037	-13.6704	8.5688	7.2645	6.8050
	Daubechies	-9.0168	10.3088	-5.9482	-7.4231	5.1840	1.2556	6.5367	16.0473	3.2873
	Meyer	-21.6741	-7.6675	-14.3456	-19.5319	-7.8079	-13.9251	14.4333	8.5846	7.4738
	Battle-Lemarié	-24.3668	-2.5878	-9.5006	-24.0618	-2.4188	-9.6513	19.0808	3.5698	4.0442
4	Shannon	-24.1532	-13.3252	-8.4140	-23.5640	-12.3671	-5.6234	16.6732	17.0482	12.6458
	Daubechies	-17.6935	3.6399	-5.4503	-15.8065	-1.1751	-5.6441	7.8687	18.0739	4.2219
	Meyer	-31.5287	-16.8619	-12.3544	-34.2040	-20.4032	-13.4978	20.9125	13.2374	10.1363
	Battle-Lemarié	-20.4299	-4.5728	-4.6614	-13.4767	-6.4869	-4.8664	17.9377	7.5612	4.2059
5	Shannon	-17.7301	-3.4409	-13.2128	-15.2637	1.7691	-9.2547	18.3200	13.0490	13.5674
	Daubechies	-17.7301	3.4409	-13.2128	-15.2637	1.7691	-9.2547	18.3200	13.0490	13.5674
	Meyer	-22.7259	-7.0328	-16.9923	-14.1805	1.0138	-8.6092	20.2726	15.7955	16.1148
	Battle-Lemarié	0.2292	15.4572	1.0347	4.1586	15.1660	-1.8023	14.3101	10.7499	5.7476
6	Shannon	-10.7263	2.5601	-6.1637	-4.9058	7.1443	-0.7929	15.9524	11.0973	11.4525
	Daubechies	-18.0931	9.1515	-0.6650	-4.9914	9.0978	-0.6859	26.1343	4.2960	0.0888
	Meyer	-12.6006	3.2345	-5.8314	-8.4605	7.2359	-0.8047	15.8750	8.4701	8.7088
	Battle-Lemarié	-1.5164	16.3486	2.1872	4.6050	17.4862	-0.4954	17.6942	10.6804	5.3346
7	Shannon	-27.9718	-23.2911	-26.5562	-30.3089	-26.1877	-33.1029	16.7338	14.7308	9.3988
	Daubechies	-5.7594	-1.0330	-13.5146	-6.6196	-1.3947	-15.2089	2.8753	3.3705	5.5848
	Meyer	-29.1160	-22.8944	-26.5143	-32.4942	-25.6679	-33.0689	17.6806	14.6209	9.6894
	Battle-Lemarié	3.2665	8.1615	1.1993	2.7752	5.0363	1.4432	1.9827	6.0327	0.7568
8	Shannon	-17.7370	-11.5636	-24.4437	-15.6891	-12.7276	-25.8691	14.2071	10.6355	15.2556
	Daubechies	0.6492	5.9883	-3.0481	0.6996	4.8684	-2.9766	0.2792	5.2747	0.2402
	Meyer	-18.6732	-11.1286	-24.6974	-19.0103	-12.2142	-26.4521	14.1214	10.0168	14.7596
	Battle-Lemarié	-3.6220	14.8301	-0.2754	5.7904	14.9184	-0.5609	22.7281	3.8925	2.0306
9	Shannon	-23.5139	-15.7896	-26.9325	-17.0989	-11.5982	-20.8504	17.0451	13.2019	16.1509
	Daubechies	-2.3425	5.2443	-1.8497	0.0373	4.6431	-1.7107	11.7608	4.5717	0.8279
	Meyer	-26.2271	-18.2931	-29.7197	-21.2537	-11.4962	-22.8726	19.2883	15.7809	18.4942
	Battle-Lemarié	-5.6343	9.1062	-0.0500	1.1396	6.9116	0.2970	18.4343	6.2239	0.8726

Table 5

What has been described here can be seen in Table 4 and 5 an Figure 29. The Shannon and the Meyer wavelet gave by far the best results for the nine differential equations. This was also reflected in other simulations with systems and examples from the reaction kinetics. In some examples the median over the logarithmic mean square error (*ln* (*mse*)) is even positive for the wavelets of Daubechies and Battle Lemarié.

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