Identifying a Superposition with Trigonometric Functions by Applying a MRA with the Shannon Wavelet

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Abstract

The multi resolution analysis (MRA) of the wavelet theory defines a sequence of close subspaces $\{V_j\}_{j\in Z}$ with $V_j \subset V_{j+1} \subset L^2(R)$. The trigonometric functions sin and cos are not quadratic integrable on *R*. However we can express them with bases functions from V_j by using the Shannon wavelet.

Introduction

In this article we use the Shannon wavelet. For the approximation using the space V_j we even can use functions that are not quadratic integrable on *R* if we only need an approximation on a finite interval *I* as in practical case. Considering the interval *I*, we could use the function $1_I \cdot f$ instead of *f*, if *f* is quadratic integrable on *I* (with the indicator function 1) and then $1_I \cdot f$ is in $\mathcal{L}^2(R)$. But that leads often to worse approximations (see [5], [6] and [7]). For trigonometric functions like sin and cos (or $e^{ia\cdot}$) we can calculate directly the bases coefficients and under certain conditions we can detect a superposition with that trigonometric functions like with the Fourier analysis.

With the scaling function ϕ of the MRA we get an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$. So we get the orthogonal projection from a $\mathcal{L}^2(R)$ function *f* in V_j with

$$f_j(t) = \sum_k f_k^j \phi_{j,k}(t) \text{ with } f_k^j = \langle f, \phi_{j,k} \rangle = \int_{-\infty}^{\infty} f(t)\cdot\overline{\phi_{j,k}(t)}dt \ .$$

In the MRA the spaces V_j are closed subspaces of $L^2(R)$. If *f* is not quadratic integrabel on *R* we say that we can "identify" *f* with V_j, instead of *f* is in V_j, if we can express *f* with the orthonormal basis of V_j.

Example:

Let be

$$f(t) = e^{-t^2} + 0.05 \cdot sin(8t) \ .$$

We show the graph of *f* together with the approximation f_1:

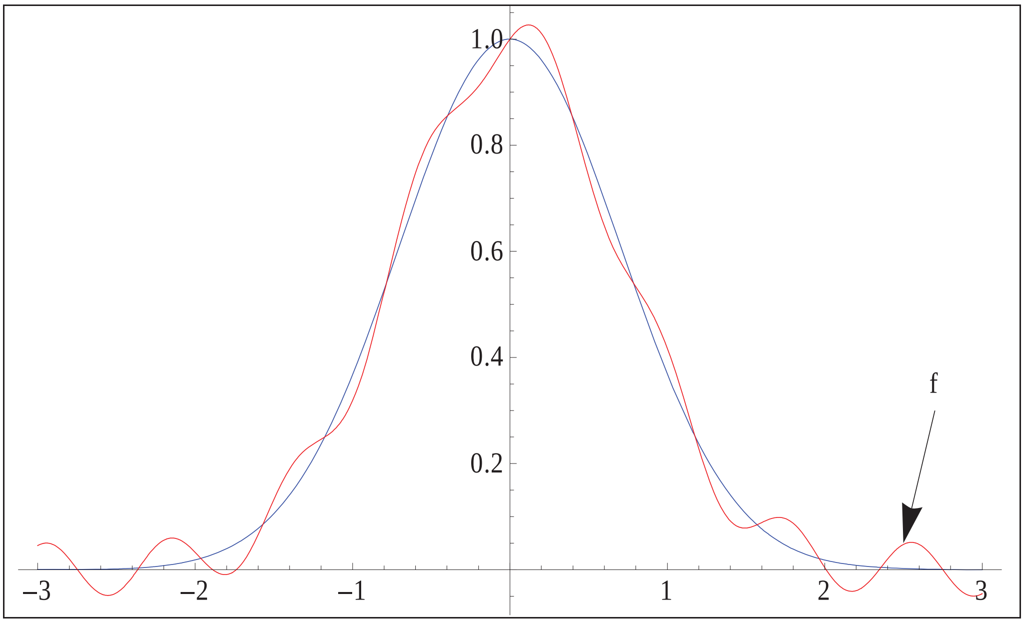

Figure 1

With the function d_1 we can “identify” the superposition term $0.05 \cdot \sin(8t)$, what we can see graphically with the graph of d_1 and soon theoretically.

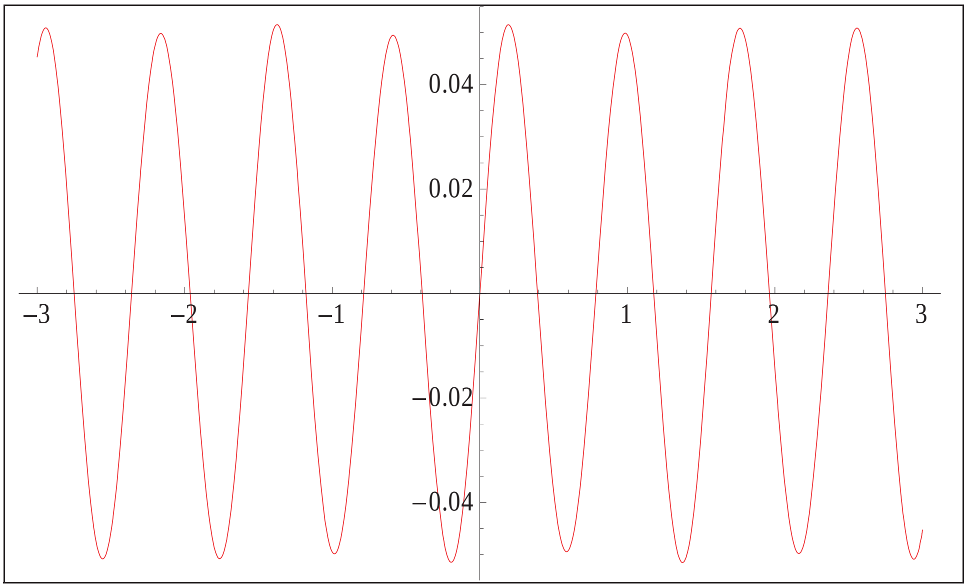

Figure 2

sin(at) and V_j

If we use the Shannon wavelet, f is in V_j if $supp\ F = [-2^j \cdot \pi, 2^j \cdot \pi]$ (or if $supp\ F \subseteq [-2^j \cdot \pi, 2^j \cdot \pi]$). If f is in detail space W_j then f is in V_{j+1} but not in V_j, because of $V_{j+1} = V_j \oplus W_j$. So if f is quadratic integrable on R then f is in W_j if $supp\ F \subset [-2^{j+1} \cdot \pi, -2^j \cdot \pi) \cup (2^j \cdot \pi, 2^{j+1} \cdot \pi]$.

In the example above we saw, that we could recognize if a function f is superposed with a sinus function, for example $f(t) = g(t) + c \cdot \sin(at)$, when we use the Shannon wavelet in a MRA.

The reason is: The Fourier transform of $h(t) = sin(at)$ is

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \cdot e^{-i\omega t} dt = i\sqrt{\pi/2} \cdot (\delta(\omega + a) - \delta(\omega - a)) ,$$

with the Dirac delta distribution δ (using for transformation and back-transformation the factor $1/\sqrt{2\pi}$). So the Fourier transform of $h(t) = sin(at)$ (from now we choose only $a > 0$) is not a function and h is not quadratic integrable on R but we could show that we get for the basis coefficients in Fourier space $<H,\Phi_{j,k}> = 2^{-j/2}h(k/2^j)$ for $a < 2^j \cdot \pi$ and we can show even directly that

$$f_k^j = <h, \phi_{j,k}> = 2^{-j/2}h(k/2^j) \text{ for } a < 2^j \cdot \pi$$

although $h \notin \mathcal{L}^2(R)$ (for $a = 2^j \cdot \pi$ all f_k^j vanish). Here we can use the equations

$$\int_{-\infty}^{\infty} e^{-i \cdot a \cdot t} \cdot \phi(t) dt = 1_{[-\pi,\pi]}(a)$$

with the indicator function 1 and

$$sin(a \cdot t) = \frac{1}{2i}(e^{i \cdot a \cdot t} - e^{-i \cdot a \cdot t}) \ .$$

We can show, that the integral above exists and so we would get $f_k^j = <h, \phi_{j,k}> = 2^{-j/2}h(k/2^j)$ for $a < 2^j \cdot \pi$.

Example:
Here are graphs of $h - h_{m,j}$ with

$$h_{m,j}(t) = \sum_{k=-m}^{m} 2^{-j/2} \cdot h(k/2^j) \cdot \phi_{j,k}(t)$$

and $h(t) = sin(at)$ ($j = 1$, $a = 4$, left for $m = 40$ and right for $m = 80$):

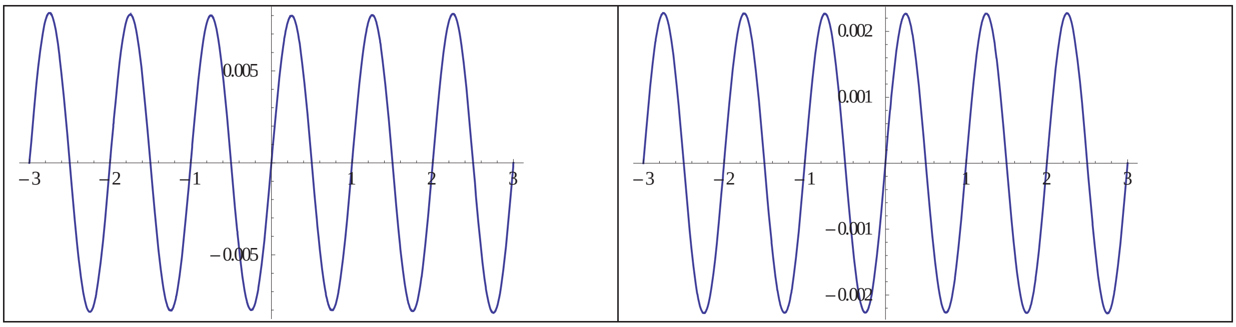

Figure 3

If we would apply the Shannon theorem on h then the condition "$\in L^1(R) \cap L^2(R)$" of the theorem is not met but we can calculate the coefficients of that sinc-series c_k and we would get $c_k = f_k^j$, too, if we set $\Omega = 2^j \cdot \pi$ (for the meaning of Ω see remark at the end of the article).

The angular frequency a must be less than $2^j \cdot \pi$ to identify h with V_j. So we could identify a superposition term $sin(at)$ (or $cos(at)$) with the detail space W_j if $2^j \cdot \pi < a < 2^{j+1} \cdot \pi$. In the first example a was 8, so we could identify the sinus term with $d_1 \in W_1$ because of $2 \cdot \pi < 8 < 4 \cdot \pi$. For the case that $a = 2^j \cdot \pi$: A superposition with $sin(at)$ could not be identified with V_j but with V_{j+1} so $sin(2^j \cdot \pi)$ could be identified with W_j.

Here are graphs (left h and $h_{m,j}$ and right $h - h_{m,j}$) for $m = 40$, $j = 0$ and $a = 1, 2, 3, 4$. We see that $sin(4t)$ could not be identified with V_0.

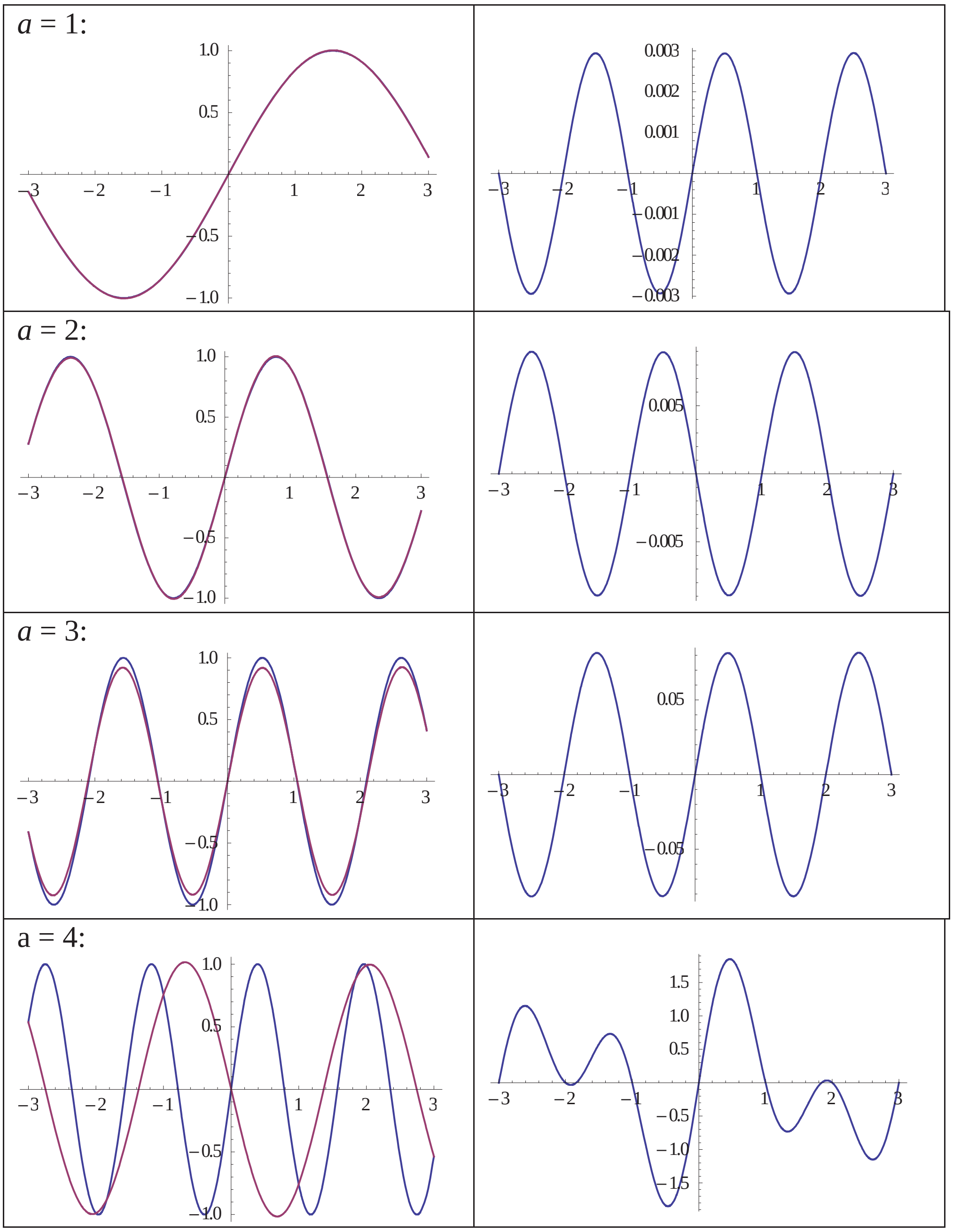

Figure 4

If a is bigger then we need a bigger m, that's what we see with the graph of $sin(3t)$. When we choose $m = 100$ then we get the following graphs:

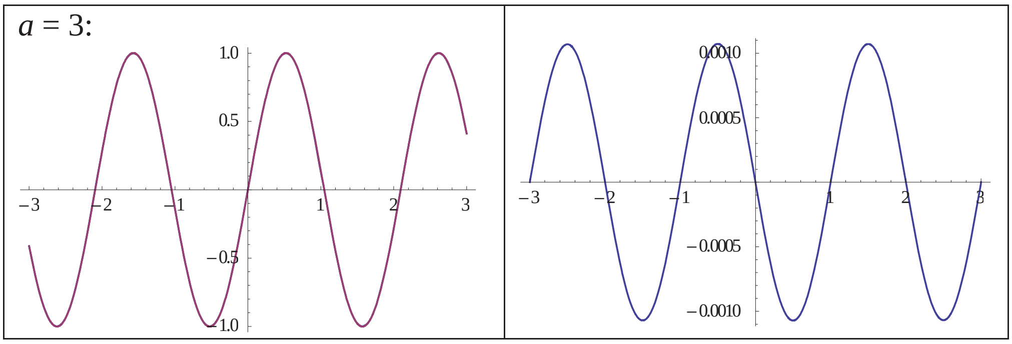

Figure 5

Here are the graphs (left h and $h_{m,j}$ and right h - $h_{m,j}$) for m = 40, j = 2 and a = 7, 8, … , 13. We see that *sin(13t)* could not be identified with V_2.

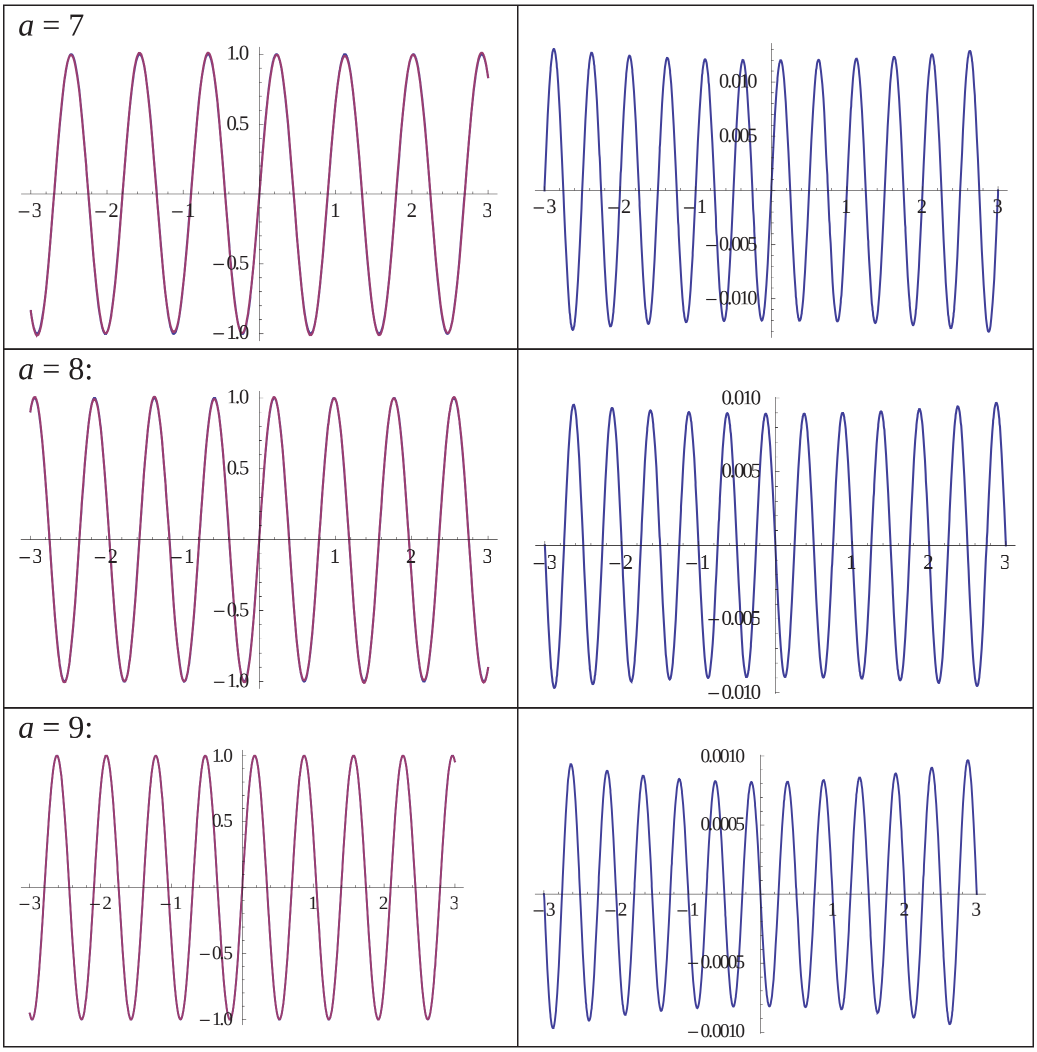

Figure 6a

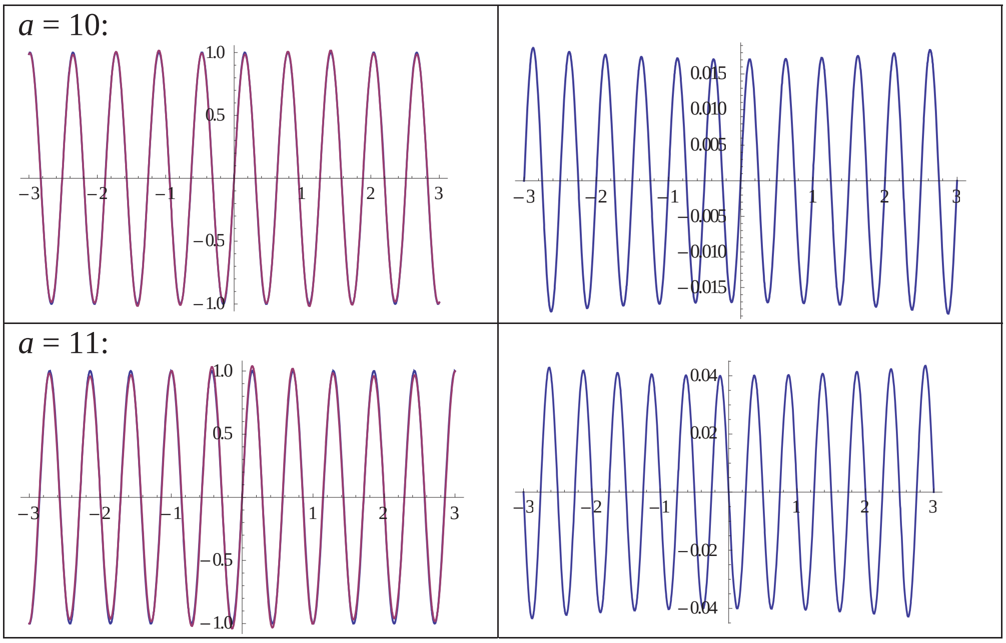

Figure 6b

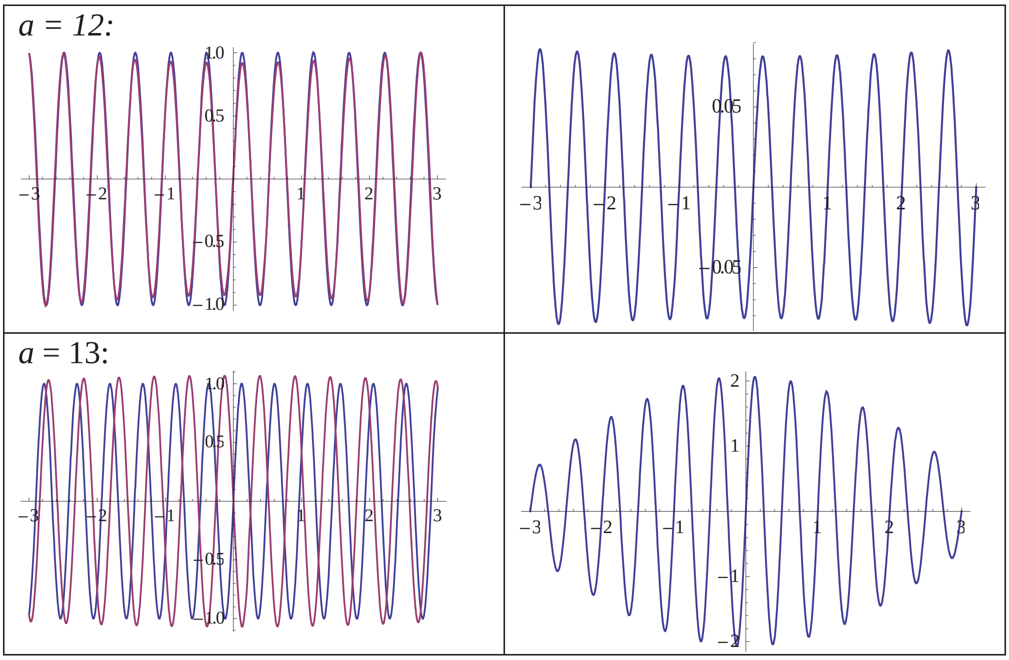

Figure 7

If we use a function of type $f(t) = g(t) + c\cdot sin(at)$ then it could be possible that we cannot identify the sinus term good in W_j also $2^j\cdot\pi \le a < 2^{j+1}\cdot\pi$ when the orthogonal projection of g in W_j has a big amount (or when the length I is too small).

Example:

For example if $f(t) = e^{-t^2}$ (which is in $\mathcal{L}^2(R)$) then the graph of d_0 is:

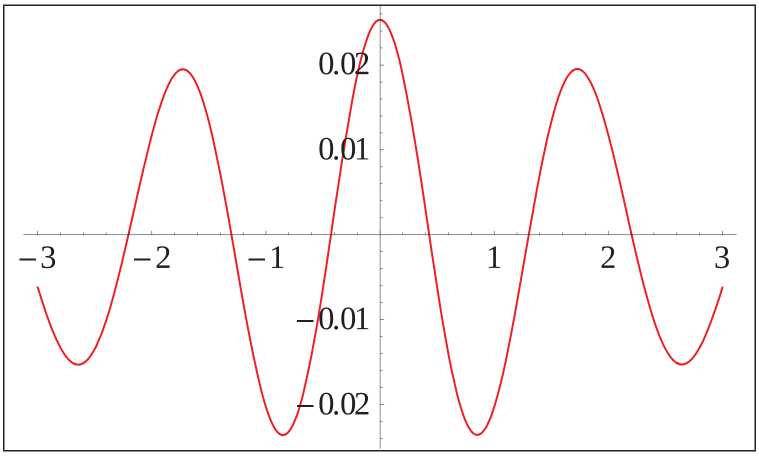

Figure 8

So the orthogonal projection d_0 of f in W_0 is not very small. f is not band-limited but the Fourier transform of f is

$$F(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

and for example $F(4\pi) \approx 5.06 \cdot 10^{-18}$. So with growing ω the function values $F(\omega)$ becomes "fast" nearly zero as well the detail functions d_j with growing j. That's what we see when we consider the approximation error in Fourier space with the difference of f and f_j (as the orthogonal projection of f in V_j). Here we could calculate the $L^2(I)$ norm $\left\| f - f_j \right\|_{L^2(I)}$ with

$$f(t) - f_j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega - \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} F(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2^j\pi} F(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{2^j\pi}^{\infty} F(\omega) e^{i\omega t} d\omega$$

if we consider the Interval I. For $I = R$ (and on R quadratic integrabel f) we get with the equation from Parseval:

$$\left\| f - f_j \right\|_{L^2} = \sqrt{\int_{-\infty}^{-2^j\pi} \left| F(\omega) \right|^2 d\omega + \int_{2^j\pi}^{\infty} \left| F(\omega) \right|^2 d\omega}$$

So we see that it is important for a good approximation with small j how "fast" $|F(\omega)|$ becomes small with increasing $|\omega|$. If the function f is continuous we could also use the maximum norm.

Analogous we get for $I = R$:

$$\left\| d_j \right\|_{L^2} = \sqrt{\int_{-2^{j+1}\pi}^{-2^j\pi} \left| F(\omega) \right|^2 d\omega + \int_{2^j\pi}^{2^{j+1}\pi} \left| F(\omega) \right|^2 d\omega}$$

Example:

Now we consider the function $f(t) = e^{-t^2} + 0.06\,sin(4t) + 0.02\,sin(10t)$, with the Graph:

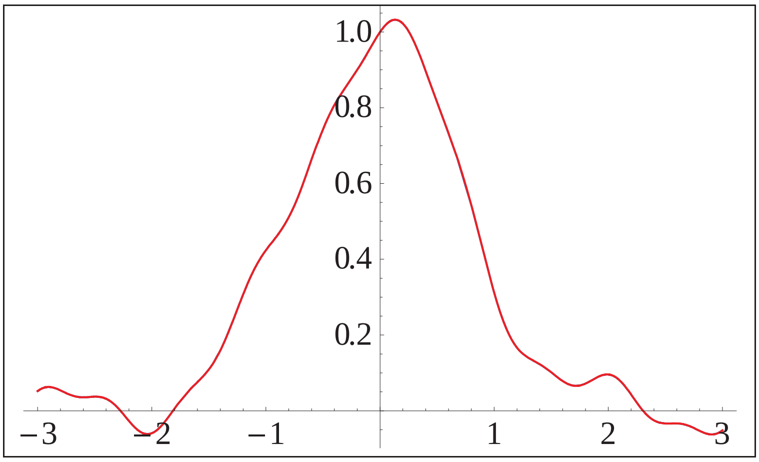

Figure 9

For the following numerical integrations in order to calculate d_j and f_j we used the interval I = [-30, 30]. We see no differences between the graph of d_1 and *0.02 sin(10t)*:

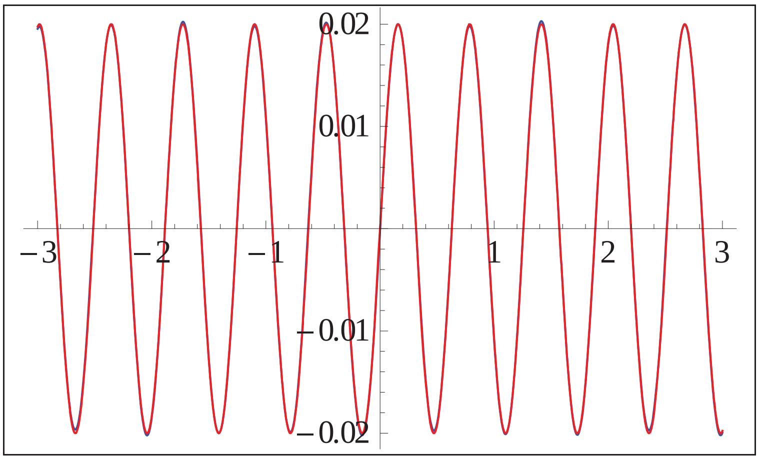

Figure 10

But between the graph of d_0 and *0.06sin(4t)* (which is dashed) we see a difference:

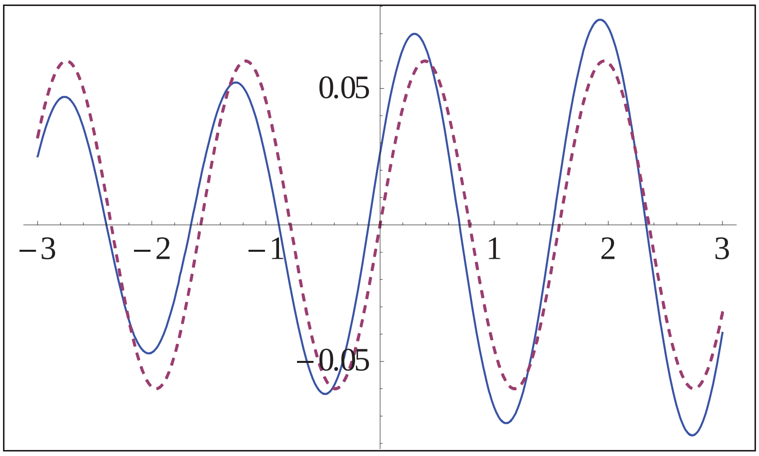

Figure 11

In d_1 the part of the orthogonal projection of e^{-t^2} does not have a big amount, but in d_0.

Here is the graph of f_1 and f (f is dashed):

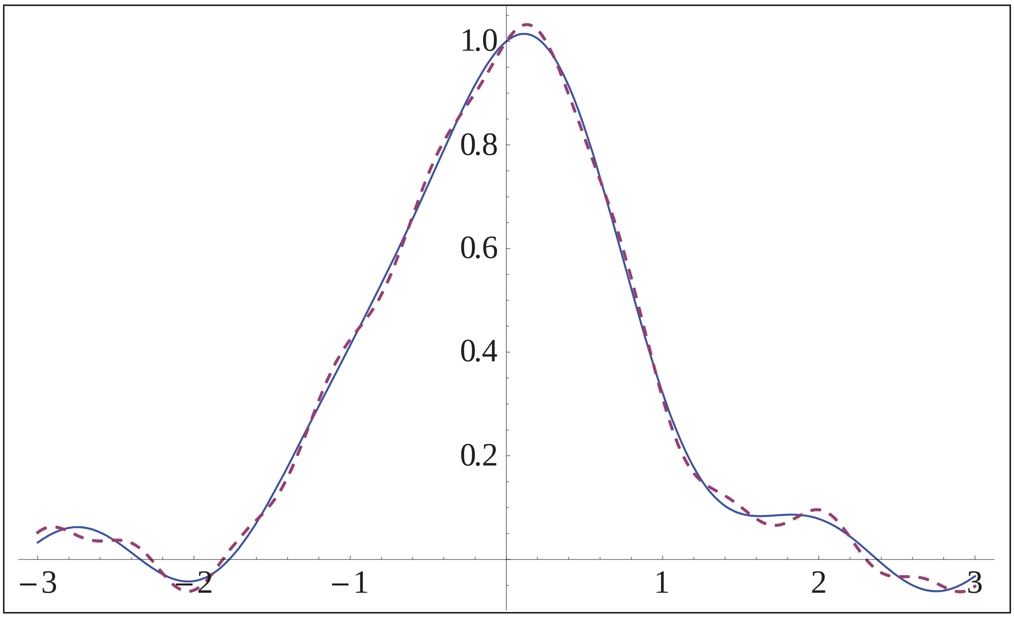

Figure 12

Here is the graph of f_0 and f (f is dashed):

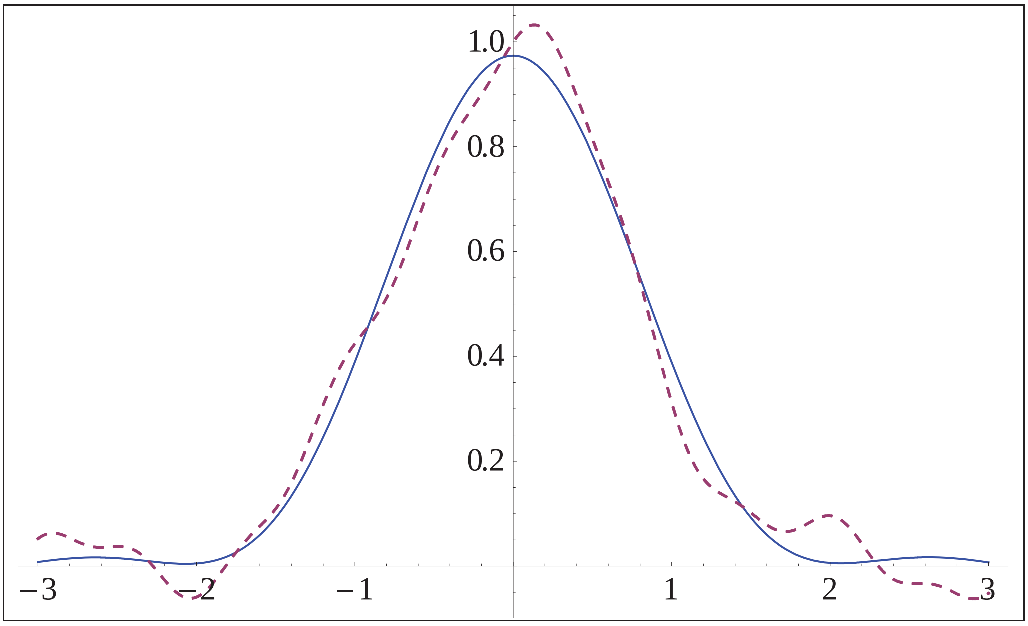

Figure 13

Finally here is the graph of the Fourier transform of $1_I(t)\cdot 0.06\sin(4t)$ divided by i (for I = [-30,30]) which is concentrated at the points $\omega = \pm 4$ (what is seen even better the bigger the interval I is):

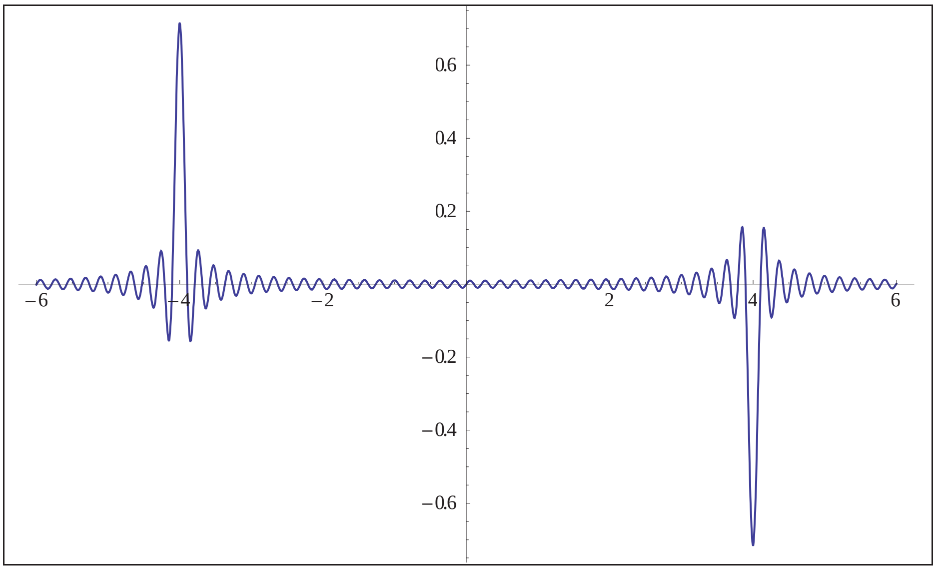

Figure 14

Remark:
We can also show with a different path (what we know from above), that for example $sin(a\cdot\pi\cdot t)$ can be expressed through the bases coefficients of V_j. Here we use the often used notation of the Shannon theorem. In V_j we have $\Omega = 2^j\cdot\pi$.

If the Fourier transform F of f has compact support ($supp\ F \subseteq [-\Omega, \Omega]$) and $f\in\mathcal{L}(R)\cap\mathcal{L}^2(R)$ than

$f(t) = f_s(t)$ (for almost all real t) with

$$f_s(t) = \sum_{k\in Z} f\left(\frac{k\cdot\pi}{\Omega}\right)\cdot\frac{sin(\Omega\cdot t - k\cdot\pi)}{\Omega\cdot t - k\cdot\pi} = \sum_{k\in Z} f\left(\frac{k\cdot\pi}{\Omega}\right)\cdot\frac{sin(\Omega\cdot t)\cdot(-1)^k}{\Omega\cdot t - k\cdot\pi}$$

$$= sin(\Omega\cdot t)\cdot\sum_{k\in Z} f\left(\frac{k\cdot\pi}{\Omega}\right)\cdot\frac{(-1)^k}{\Omega\cdot t - k\cdot\pi}$$

That’s Shannon’s theorem.

We consider $f(t) = sin(a\cdot\pi\cdot t)$ and we set $\Omega = 2\cdot a\cdot\pi$. If we set $\Omega = a\cdot\pi$, we would get 0. We could choose other $\Omega > a\cdot\pi$, but for $\Omega = 2\cdot a\cdot\pi$ we see easily that f can be expressed with the Shannon series, even f is not in $\mathcal{L}(R)\cap\mathcal{L}^2(R)$, what is an assumption of the Shannon theorem. With that choice of Ω the coefficients $f\left(\frac{k\cdot\pi}{\Omega}\right) \in \{-1, 0, 1\}$.

$$f_s(t) = sin(2\cdot a\cdot\pi\cdot t)\cdot\sum_{k\in Z} sin\left(a\cdot\pi\cdot\frac{k\cdot\pi}{2\cdot a\cdot\pi}\right)\cdot\frac{(-1)^k}{2\cdot a\cdot\pi\cdot t - k\cdot\pi}$$

$$= \frac{sin(2\cdot a\cdot\pi\cdot t)}{\pi}\cdot\sum_{k\in Z} sin\left(\frac{k\cdot\pi}{2}\right)\cdot\frac{(-1)^k}{2\cdot a\cdot t - k}$$

Here is:

$$sin\left(\frac{k\cdot\pi}{2}\right)=\begin{cases}0 & if \quad k \; is \; even\\ sign(k) & if \quad |k|=1,5,9,...\\ -sign(k) & if \quad |k|=3,7,11,...\end{cases}$$

So we get:

$$f_s(t)=\frac{sin(2\cdot a\cdot\pi\cdot t)}{\pi}\cdot\sum_{k\in Z}\frac{(-1)^{k+1}}{2\cdot a\cdot t-(2k+1)}$$

$$=\frac{sin(2\cdot a\cdot\pi\cdot t)}{\pi}\cdot 2\cdot\underbrace{\sum_{k\in N_0}\frac{(-1)^{k+1}\cdot(2k+1)}{(2k+1)^2-(2\cdot a\cdot t)^2}}_{=\pi/2\cdot sec(a\cdot\pi\cdot t)}$$

$$=sin(2\cdot a\cdot\pi\cdot t)\cdot sec(a\cdot\pi\cdot t)\cdot 1/2$$

$$=sin(a\cdot\pi\cdot t)=f(t)$$

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