Journal of Approximation Theory and Applied Mathematics

An Approximation on a Compact Interval Calculated with a Wavelet Collocation Method can Lead to Much Better Results than other Methods

M. Schuchmann and M. Rasguljajew from the Darmstadt University of Applied Sciences

Abstract

As part of a research project we ran several simulations with a wavelet collocation method to find out how the optimal parameters can be determined. Comparing the approximations of functions on a compact interval I, we noticed that when y is not in $L^2(R)$ a certain wavelet collocation method approximation was significantly better than projecting $1_I y$ orthogonal to V_j (with the indicator function 1_I). This method even gives very good approximations when using relatively few basis elements.

Introduction

In the wavelet theory a scaling function ϕ is used, which belongs to a MSA (multi scale analysis). From the MSA we know, that we can construct an orthonormal basis of a closed subspace V_j, where V_j belongs to a the sequence of subspaces with the following property:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k\in Z}$ is an orthonormal basis of V_j with $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$.

We use the following approximation function

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t) \quad \text{, with } \phi\in C^1(R).$$

k_{max} and k_{min} depend on the approximation interval $[t_0, t_{end}]$ (see [7]).

Now we can approximate the solution of an initial value problem $y' = f(y,t)$ and $y(t_0) = y_0$ by minimization of the following function

$$(1) \qquad Q(c) = \sum_{i=1}^{m} \left\| y_j'(t_i) - f(y_j(t_i), t_i) \right\|_2^2 + \left\| y_j(t_0) - y_0 \right\|_2^2 \quad .$$

For $m = |k_{max} - k_{min}|$ we get an equivalent problem:

$$y_j'(t_i) = f(y_j(t_i), t_i) \text{ for } i = 1, 2, ..., m \text{ and } y_j(t_0) = y_0 .$$

Analogous we could treat boundary conditions instead of the initial condition. This method can be even used analogous for PDEs, ODEs of higher order or ODEs, which have the Form $F(y', y, t) = 0$.

Error Estimation

For the orthogonal projection y_j from y in V_j we know from the Gilbert-Strang Theory (see [9]) an upper bound of the approximation error in dependency of the order p: If the wavelet is of order p then the approximation error has the order $O(2^{-jp})$ if $\|y^{(p)}\|_{L^2} < \infty$ and

$$\|y - y_j\|_{L^2} \le C_\phi \cdot 2^{-jp} \cdot \|y^{(p)}\|_{L^2} \quad .$$

If a wavelet is of order p the scaling function ϕ even has a interpolation property, because then we can construct the functions t^r with $r = 0, 1, ..., p-1$ over a linear combination of $\phi(t-k)$ (see [9]). That's also a property of the so called interpolating wavelets. For interpolating wavelets we find error estimations in [5] and [8].

In the examples we used the Shannon wavelet. For this wavelet we have additional information about the error in the Fourier space from the Shannon theorem (under the conditions of this theorem). For a good approximation with a small j the behavior of $Y(\omega)$ with growing $|\omega|$ is important, because

$$y(t) - y_j(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega - \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} Y(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2^j\pi} Y(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{2^j\pi}^{\infty} Y(\omega) e^{i\omega t} d\omega \quad .$$

With the Parseval theorem we get

$$\|y - y_j\|_{L^2} = \sqrt{\int_{-\infty}^{-2^j\pi} |Y(\omega)|^2 d\omega + \int_{2^j\pi}^{\infty} |Y(\omega)|^2 d\omega} \quad .$$

But if we calculate y_j over the minimization of Q we generally don't get a orthogonal projection form y in V_j and generally y is not quadratic integrable over R. There we can use the following theorem:

Theorem 1:
Assumptions: We have a initial value problem $y' = f(y, t)$ with $y(t_0) = y_0$ and

$$||y_j(t_0) - y(t_0)|| \le \delta \ ,$$

$$(4) \quad ||y_j'(t) - f(y_j(t), t)|| \le M$$

and

$$(5) \quad ||f(y(t), t) - f(y_j(t), t)|| \le L \cdot ||y(t) - y_j(t)|| \text{ with } L > 0.$$

Then we get for $t \ge t_0$:

$$||y_j(t) - y(t)|| \le \delta \cdot e^{L(t-t_0)} + M/L \cdot (e^{L(t-t_0)} - 1)$$

For a proof see [6].

In the examples and in many simulations we saw that δ was very small. If we assume $\delta = 0$ then we get the following error estimation (if we consider the compact interval $I = [t_0, t_{end}]$ and under the assumptions of theorem 1):

$$\|y_j - y\|_{L^2(I)} \le M/L \cdot \sqrt{\int_{t_0}^{t_{end}} (e^{L(t-t_0)} - 1)^2 \, dt}$$
$$\le M/L \cdot \sqrt{1/2L \cdot \left(-4e^{L(t_{end}-t_0)} + 3 + e^{2L(t_{end}-t_0)}\right) + t_{end} - t_0}$$

So if M is very small then we can get very good approximations if f is Lipschitz continuous.

Comparing the Two Ways of Approximation

Now we want to approximate two functions in the following two examples, which are not quadratic integrable on R.

Example 1:
We begin with an approximation of the function $y(t) = e^{-t}$ on $I = [0, 1]$. y is not in $L^2(R)$, but every on I continuous function is in $L^2(I)$ or $1_I y$ (with indicator function 1_I of I) is in $L^2(R)$. So we set $k_{max} = -k_{min} = 20$ and we calculate an approximation function by an orthogonal projection from $1_I y$ on V_1. Therefore we calculate the coefficients of the approximation function over a scalar product:

$$c_k = \langle 1_{[0,1]} y, \phi_{j,k} \rangle = \int_0^1 y(t) \cdot \phi_{1,k}(t) dt$$

With the Shannon wavelet we get a worse approximation (dashed line is thee graph of y):

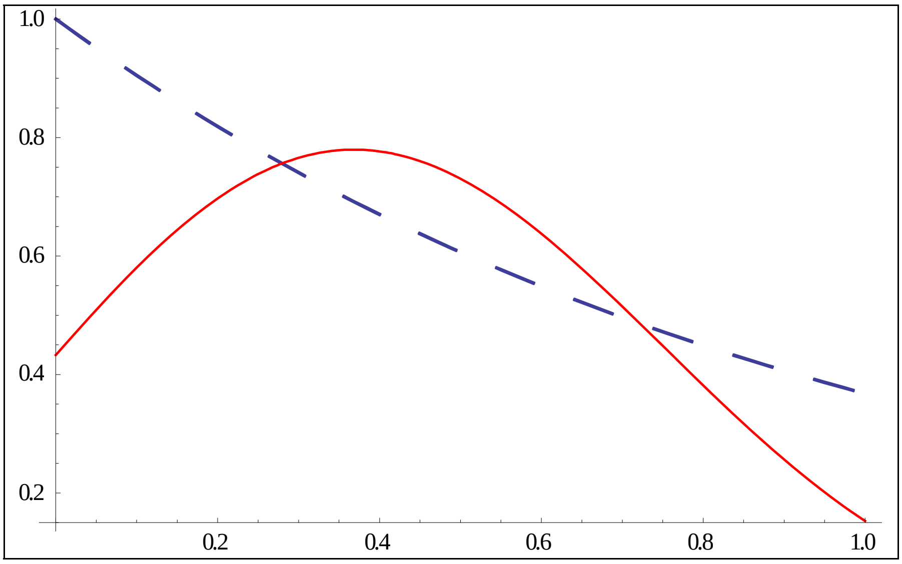

Figure 1. Graphs of y_1 (orthogonal projection form $I_I y$ on V_1) and y

With the Daubechies wavelet of order 8 we get no better approximation:

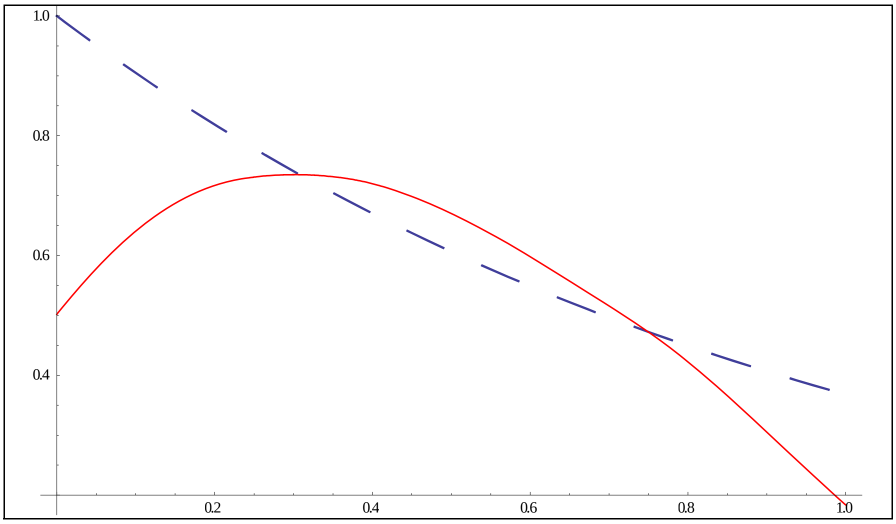

Figure 2. Graphs of y_1 (orthogonal projection form $I_I y$ on V_1) and y, Daubechies wavelet order 8

Even if we set $j = 3$ and $k_{max} =- k_{min} = 24$ we get not really a useful approximation:

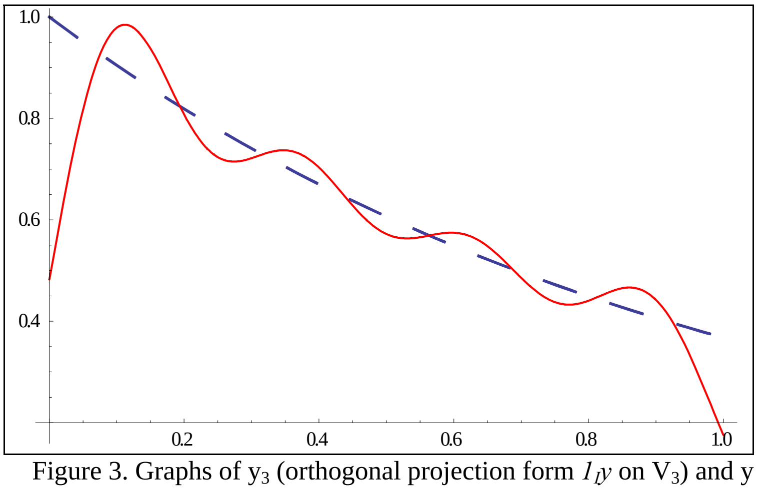

Figure 3. Graphs of y_3 (orthogonal projection form $I_I y$ on V_3) and y

If we take a look on the graphs on a bigger interval we see, that we calculated the best approximation of the function $1_{[0,1]}\cdot y$. That function is on I identically to y and outside I equal to zero:

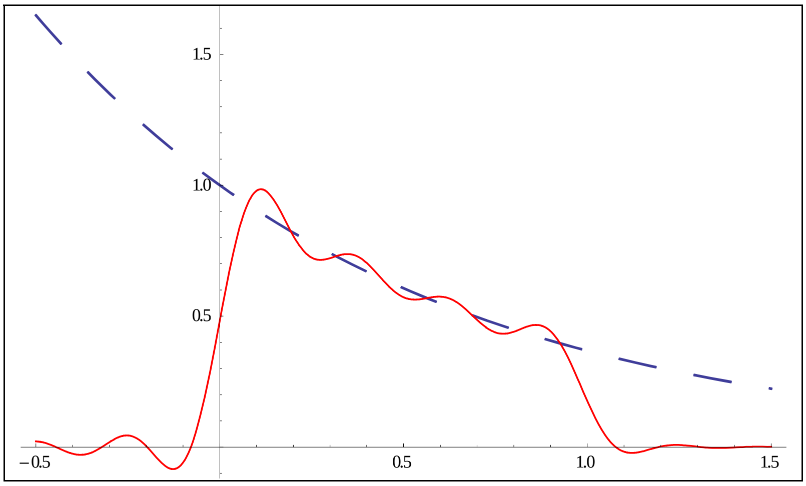

Figure 4. Graphs of y_3 (orthogonal projection form $1_I y$ on V_3) and y, bigger area

For a comparison the same approximation with the Daubechies wavelet of order 8:

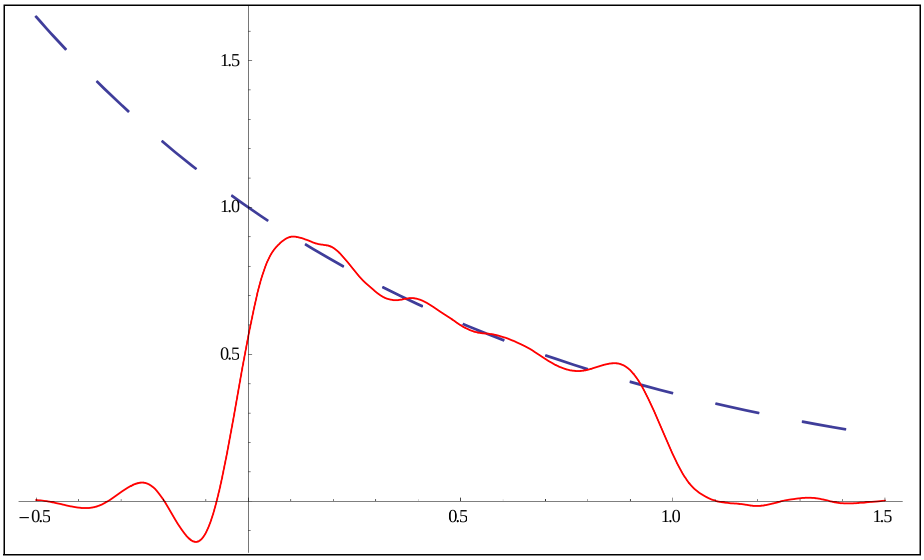

Figure 5. Graph of y_3 (orthogonal projection form $1_I y$ on V3) and y,
Daubechies wavelet and bigger area

So the orhogonal projection considers the whole real axis, when we integrate only over I.

Now we calculate the coeffictions c_k by the minimization of Q (see (1)). We use the initial value problem $y' = -y$, $y(0) = 1$ and set an even smaller summation area with $k_{min} = -8$ and $k_{max} = 10$ and $j = 1$. We use the collocation points $t_i = i/20$ with $i = 0, 2, ..., 20$.

Graphically we see no difference between the approximation function y_j and y on I ($Q_{min} \approx 3.00724 \cdot 10^{-30}$):

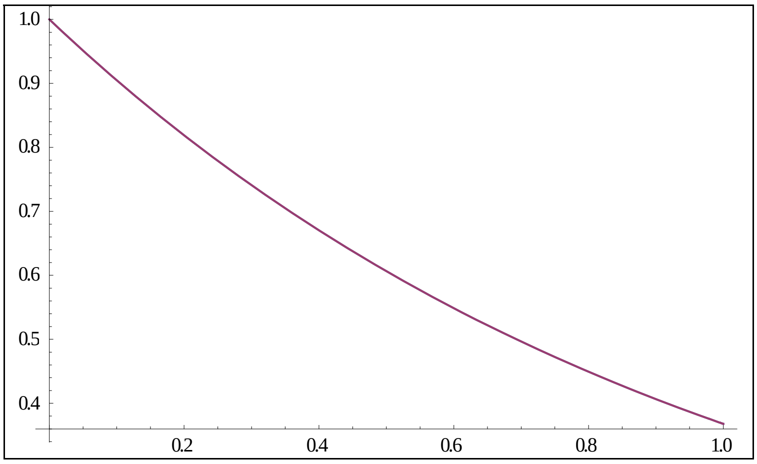

Figure 6. Graphs of y_1 (calculated by min Q) and y

Here is the graph of the difference function $y_j - y$:

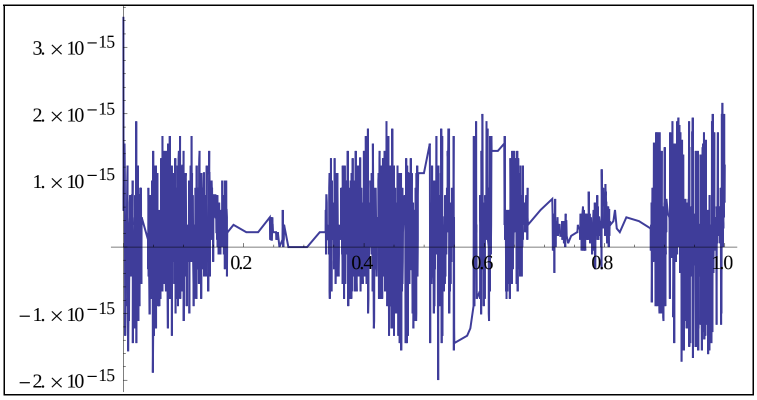

Figure 7. Graph of y_1 - y (y_1 calculated by min Q) and y

We could even use this approximation function for an extrapolation on a bigger interval than *I*. Here we see the graph of $y_1 - y$ on the interval $[-0.5, 1.5]$:

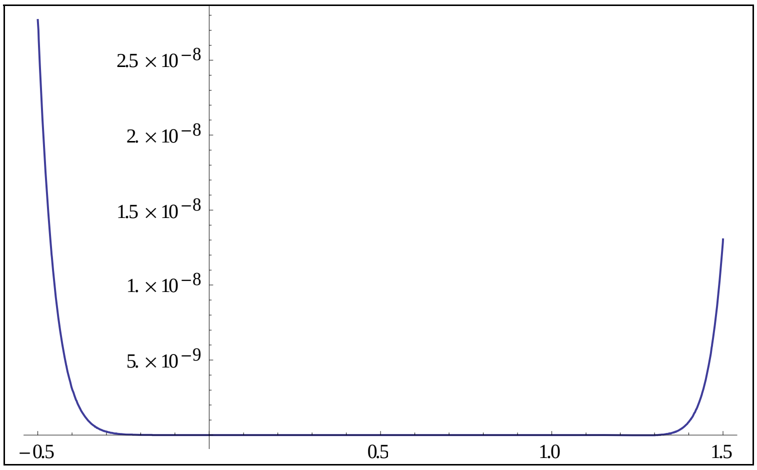

Figure 8. Graph of y_1 - y (y_1 calculated by min Q) and y on a bigger area

Here is the graph of *d* with $d(t) = (y_j(t)-f(y_j(t),t))^2$:

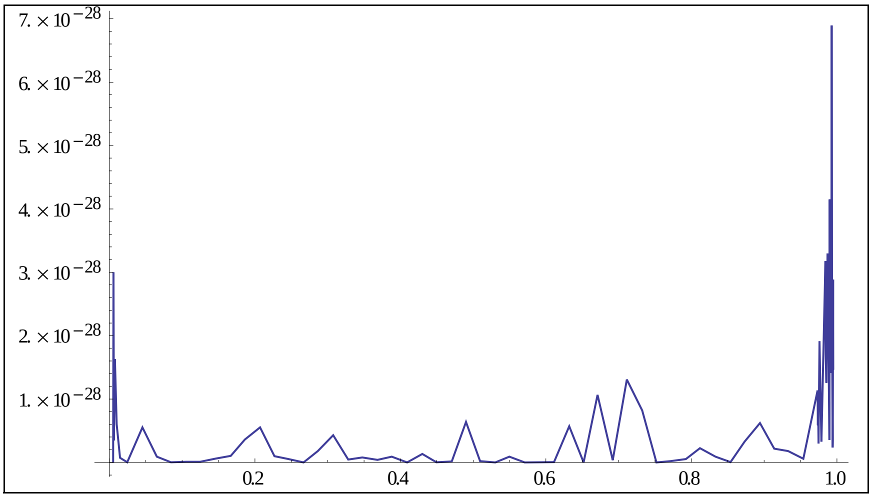

Figure 9. Graph of d

So M is small.

Example 2:
Now we consider the function $y(t) = sin(t)$, which is not quadratic integrable on $\mathbb{R}$ but we could construct y with functions out of V_j by using the Shannon wavelet.

If we use the Shannon wavelet, $y \in L^1(R) \cap L^2(R)$ is in V_j if $supp\ Y \subseteq [-2^j \cdot \pi, 2^j \cdot \pi]$ (where Y is the Fourier transform of y).

If we consider the function $h(t) = sin(at)$ then

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \cdot e^{-i\omega t} dt = i\sqrt{\pi/2} \cdot \left(\delta(\omega + a) - \delta(\omega - a)\right),$$

with the Dirac delta distribution δ (using for transformation and back-transformation the factor $1/\sqrt{2\pi}$). So the Fourier transform of $h(t) = sin(at)$ (from now we choose only $a > 0$) is not a function and h is not quadratic integrable on R but we can show that we get for the basis coefficients in Fourier space $<H, \Phi_{j,k}> = 2^{-j/2} h(k/2^j)$ for $a < 2^j \cdot \pi$.

So we set $c_k = 2^{-j/2} y(k/2^j)$ and we get for $k_{max} = -k_{min} = 15$ the following graph of $y_0 - y$:

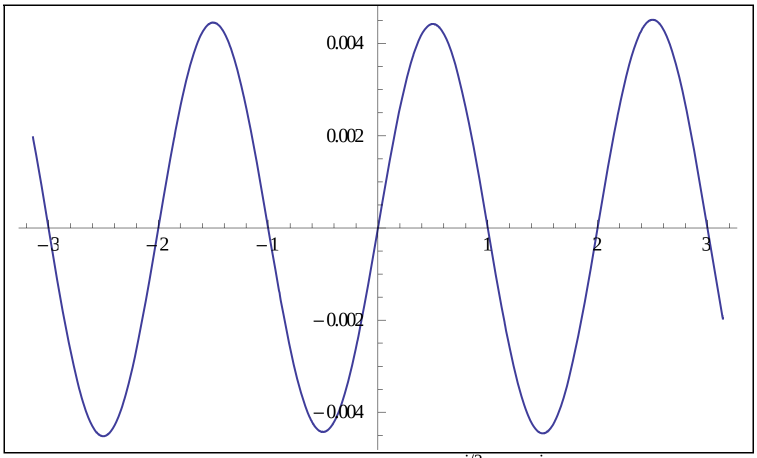

Figure 10. Graph of y_0 - y, k_{max} = 15, $c_k = 2^{-j/2}y(k/2^j)$, Shannon wavelet

For k_{max} = - k_{min} = 50:

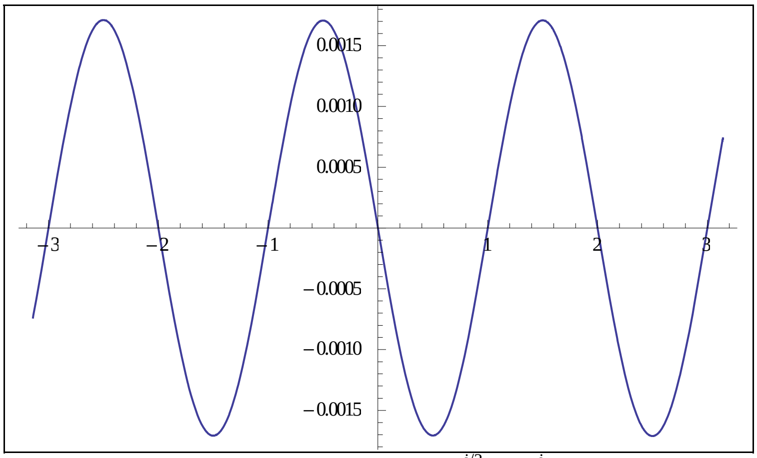

Figure 11. Graph of y_0 - y, k_{max} = 50, $c_k = 2^{-j/2}y(k/2^j)$, Shannon wavelet

For k_{max} = - k_{min} = 100:

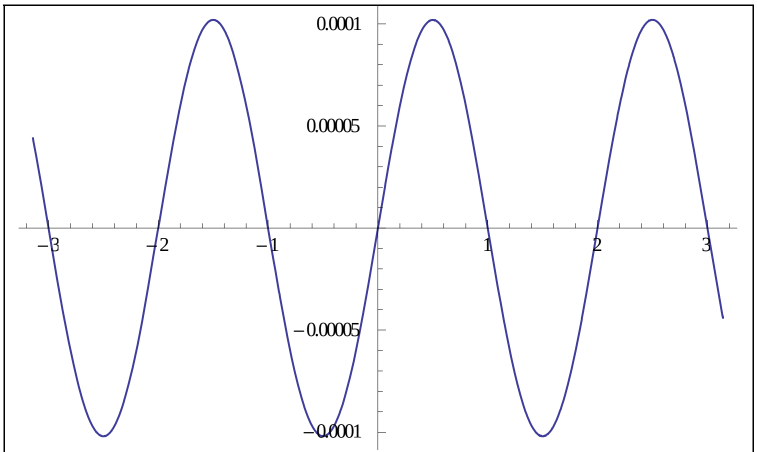

Figure 12. Graph of y_0 - y, k_{max} = 100, $c_k = 2^{-j/2}y(k/2^j)$, Shannon wavelet

If we would use the same method like in example 1 to get an approximation from y on $I = [-\pi, \pi]$ and if we calculate $c_k = \langle 1_{[-5,5]}\, y, \phi_{k,j} \rangle$, so we get with $k_{min} = -k_{max} = 15$ and $j = 0$ a bigger error $y_0 - y$:

With the Shannon wavelet, we get the following difference $y_0 - y$:

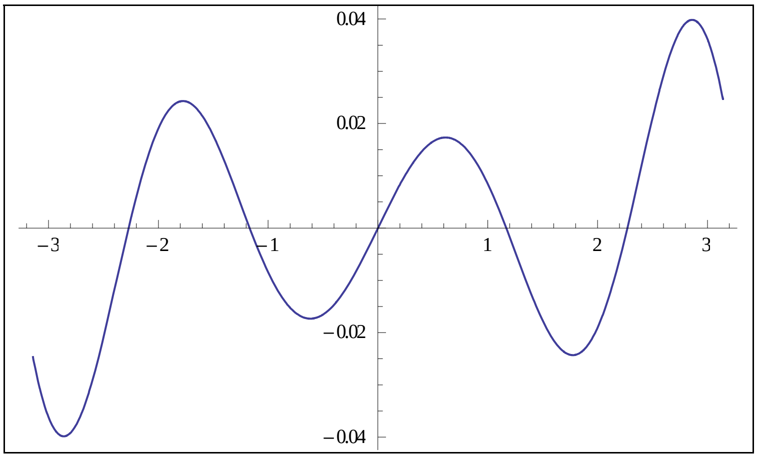

Figure 13. Graph of y_0 - y (y_0 orthogonal projection from $1_{[-5,5]}\,y$ on V_0)

Here is the problem, that the Fourier transform of y has a compact support, but not the Fourier Transform of $1_{[-5,5]}y$. Here ist the graph of magnitude spectrum of the Fourier transform of $1_{[-5,5]}y$:

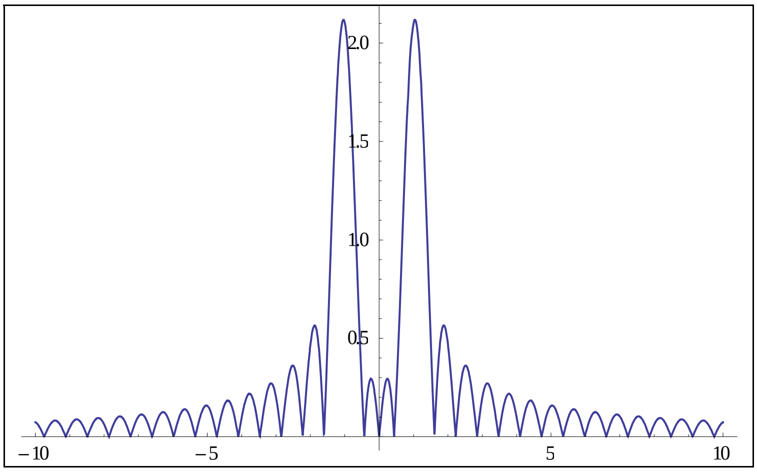

Figure 14. Magnitude spectrum of the Fourier transform from $1_{[-5,5]}\,y$

Now we consider the initial value problem with the solution $y(t) = sin(t)$:

$y'(t) = cos(t),\ y(0) = 0.$

We calculate the coefficients c_k by minimization of Q using the collocation points $t_j = -\pi + i \cdot \pi/15$, i = 0, 1, ..., 30. We set $j = 0$, $k_{max} = -k_{min} = 15$ an get the following difference $y_0 - y$ (with $Q_{min} \approx 4.7488 \cdot 10^{-28}$):

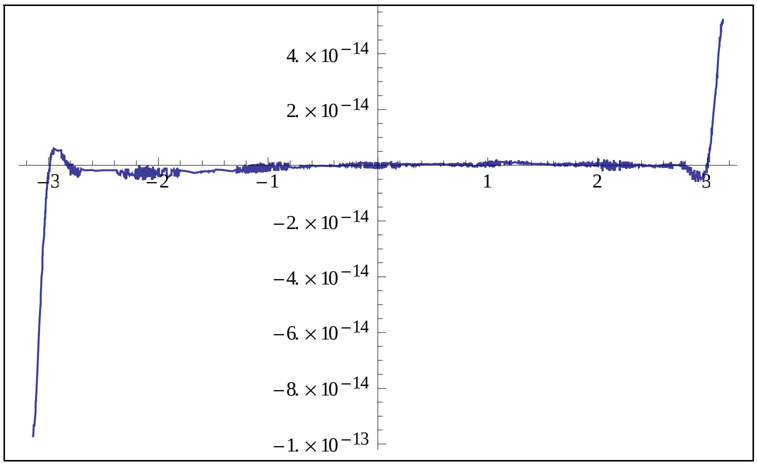

Figure 15. Graph of y_0 - y (y_0 calculated by min Q)

Conclusions

If we use a wavelet basis for the approximation of a not quadratic integrablen function *y* on a compact interval *I*, the calculation of an approximation function over the orthogonal projection form $1_I y$ on V_j can lead to a worse approximation. But if we solve numerically an initial value problem with the solution *y* by using a wavelet collocation method, we can get much better approximations. Even if *y* would be band limited generally $1_I y$ is not band limited (because of the Heisenberg uncertainty principle in the Fourier transform)..

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